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INVESTMENTS AND UNSTABILITY

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The authors of this scientific work have substantiated the criteria of measurement of risk of securities – square of profitability rate of change. The principle of “minimum risk – maximum utility” has been suggested for the consumer’s choice.
The monograph is intended for scientists and stockjobbers.

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EDITOR'S NOTE

"Economics is an art,
not a science"
Michel Condecu

These words are the tragedy of the contemporary mankind. The accidentally spoken phrase of Mr. Condecu (it has not been scientifically proved) is a direct proof of the fact that the IMF is governed not by economists, but by politicians. As, without simulation of the economic system of this or that country, without necessary calculations and forecasts on their basis, they are still ambitiously requesting execution of only their instructions.

Understanding of the real course of events, in distinction from the hypothetical equilibrium, poses many problems that have not received their proper estimation. The reason for their arising is in the fact that the participants of the events adopt decisions based on their inherent and imperfect understanding of their situation.

Assertion of complete knowledge is rather disputable, as perception of the situation by that or this participant does not mean realization of the perceived, and thus, cannot be considered as knowledge. Imperfection of understanding of the situation by its participants is a concept that is hard to determine and even harder to work with as we receive a rather conditional information that does not suggest objective knowledge. Availability of such information does not allow employing mathematical apparatus with the help of which it is possible to describe the processes of decision-making minimization.

Many well-known economists have contributed in to the modern theory of financing and investments. Among them are J.Arrow Kennet, G. Debreu, Irving Fisher, V.V. Kovalev, Z.A. Sabov, Jack Hirshleifer, R.B. Tyan, V.F. Zalunin, H.M. Markowitz, M.A. Golzberg, R.C. Merton, M. Miller, I.L. Sazonets, F. Modigliani, A.V. Mertnes, Oscar Morgenstern, G. Von Neuman, Ya. K. Bersutsky, W.F. Sharpe, K.F. Kovalchuk, G. Hicks, G.M. Keins, A.I. Yastremsky and James Tobin, nine of whom are Nobel prize winners in economics and this list is far from completion.

The least action principle (that proved good in solving physical problems) has been undertaken to solve problems of holding of the optimal investment portfolio.

In majority of economic situations, in which participants are operating with the help of scientific methods, one set of conditions follows the other, independent of somebody's conceptions in this regard.

Phenomena which are studied by the humanities, including financial and economic activity, especially under uncertainty are inherent to the market relations, involve highly intellectual participants, and all this makes the process of decision-making even more difficult due to the natural bias of opinions. Instead of the direct line connecting one set of conditions with another, we are constantly facing transition from objective conditions that can be controlled and observed to the subconscious observations of participants: participants of situation phenomena are relying on not objective conditions in their decision-making, but on their intuition that interprets these conditions in a certain way. This is a very important moment with far-reaching consequences.

This postulate introduces the element of uncertainty of economic phenomena that make the process of decision-making in economics less subject to generalization, prediction and explanation. That is why this element of uncertainty is so destructive. Social sciences as a whole and economic theory in particular, without available mechanism, tried to make all possible to exclude or to ignore it.

Doctor of Economic Sciences V.A. Tkachenko

AUTHORS' NOTE

...only those are not mistaken, who do nothing.
Vladimir Lenin

To be disapproved –it is nothing to be afraid of;
One should be afraid of being misunderstood.
Immanuel Kant

The majority of works on investments determine the investment risk of the investment portfolio as variability of profitability that is estimated by the standard deviation (variance) of the profitability portfolio distribution.

However, this approach has a number of drawbacks, namely:

1) Existence of the quadratic form of the utility function;
2) Abidance by normal distribution of asset profitability.

According to the empirical evidence, profitability of some securities (for example, options) is not normally distributed.

This work suggests the new criteria of risk measurement – **square of profitability rate of change of securities**. It eliminates the above traditional approach drawbacks.

Each model has its own scope of application: in one case, variance will be a more precise method of the portfolio risk measurement, in others – square of profitability rate of change.

In section three, the authors tried to use the mathematical apparatus of calculus of variations in solving problems of optimal investment portfolio formation.

We hope that approaches suggested in this monograph will fine their further development and practical application.

We thank Doctor of Economic Sciences, V.A. Tkachenko for his constructive assistance in writing of this work.

COMPARATIVE CHARACTERISTIC OF THE EXISTING THEORIES AND THE THEORIES SUGGESTED BY THE AUTHORS

	EXISTING THEORIES	SUGGESTED THEORIES
1	**Equilibrium price** – the price, at which the demand is equal to the supply.	**Equilibrium price** – the price, at which probability of the goods purchase is equal to probability of sale.
2	An individual under risky conditions **make his choice depending on the maximization of the expected utility** of the result: one chooses such distribution P* from the set of alternatives that $V(P^*) = \max_P V(P) = \max_P E_p[u(\omega)]$, **where** $u(\omega)$ - **utility function. (Theory of the expected utility)**	An individual, from the set of alternatives, **is guided by the principle "minimum risk – maximum utility"**, i.e. minimum of the functional $S = \int_{t_1}^{t_2} (R - U)dt \rightarrow \min$ at $t_1 < t < t_2$, where, R – risk function, U – utility function. Within the period of time $[t_1, t_2]$, the individual distributes available resources to make the functional accept the minimum value.
3	Risk of the securities portfolio is determined by **the standard deviation** σ_n of its profitability.	Risk of the securities portfolio is determined by **the square of the portfolio profitability rate of change** $\left(\frac{dr_n}{dt}\right)^2$.
4	The measure of interrelation between two securities is **covariation of their profitability** σ_{12}.	The measure of interrelation between two securities is **the scalar product of their profitability rate of chang** $\left(\frac{dr_1}{dt}, \frac{dr_2}{dt}\right)$.

Comparative characteristic of the existing theories and the theories suggested by the authors

	EXISTING THEORIES	SUGGESTED THEORIES
5	Relative measure of interreation between profitability of two securities is their **correlation** $\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2}$.	Relative measure of interrelation between profitability of two securities is **the cosine of the averaged angle between profitability rate of change vectors** $\cos\angle(\frac{dr_1}{dt},\frac{dr_2}{dt}) = \frac{\frac{dr_1}{dt}\cdot\frac{dr_2}{dt}}{\left\Vert\frac{dr_1}{dt}\right\Vert\cdot\left\Vert\frac{dr_2}{dt}\right\Vert}$.
6	Cost of the j-securities $p(X_j) = \frac{E[X_j] - \frac{E[r_m]-r_f}{Var[r_m]}\cdot Cov[X_j, r_m]}{1+r_f}$, where, r_m – profitability of the market, r_f – risk-free rate of profitability.	Cost of the j-securities $p(X_j) = \frac{E[X_j] - \frac{E[r_m]-r_f}{\left(\frac{dr_m}{dt}\right)^2}\cdot\left(\frac{dX_j}{dt},\frac{dr_m}{dt}\right)}{1+r_f}$, where, r_m – profitability of the market, r_f – risk-free rate of profitability.
7	Systematic risk (beta) is equal to: $\beta_j = \frac{Cov[r_j, r_m]}{Var[r_m]}$.	Systematic risk (tau) is equal to: $\tau_j = \frac{\left(\frac{dr_j}{dt},\frac{dr_m}{dt}\right)}{\left(\frac{dr_m}{dt}\right)^2}$.
8	Markovitz's model $\min_{x_1,x_2,\ldots,x_n}\{\sigma_p\} = \min_{x_1,x_2,\ldots,x_n}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}\right\}$.	Markovitz's model $\min_{x_1,x_2,\ldots,x_n}\left\{\left(\frac{dr_p}{dt}\right)^2\right\} = \min_{x_1,x_2,\ldots,x_n}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{dr_i}{dt}\frac{dr_j}{dt}\right\}$.
9	No analogue	Dynamics of the securities profitability is described by the wave function $\Psi(r)$.

Comparative characteristic of the existing theories and the theories suggested by the authors

	EXISTING THEORIES	SUGGESTED THEORIES
10	No analogue	Each economic variable is compared with a certain linear operator $\hat{F}$.
11	No analogue	**Principle of superposition.** If the economic system can be in the states that are described by functions $\Psi_1, \Psi_2, ..., \Psi_n$, then it may be in the complex state $\Psi = c_1\Psi_1 + c_2\Psi_2 + ... + c_n\Psi_n$, where, $c_1, c_2, ..., c_n$ - certain are arbitrary complex numbers.
12	No analogue	The only possible results of observation of the economic dynamic variable in the given state are the eigenvalues of the compared operator $\hat{F}$.
13	No analogue	Probability $W_\Psi(f)$ received during the change of the dynamic economic variable of the value *f* in the state $\|\Psi\rangle$ is set by the formula: $W_\Psi(f) = \|\langle f \| \Psi\rangle\|^2$, where, $\langle f \|$ - eigenvector of the operator $\hat{F}$, that belongs to the eigenvalue *f*. The function $\|\Psi\rangle$ is normalized to one.

Comparative characteristic of the existing theories and the theories suggested by the authors

	EXISTING THEORIES	SUGGESTED THEORIES
14	No analogue	Portfolio of securities with minimum risk, which is described by the wave function $\Psi = c_1\Psi_1 + c_2\Psi_2 + ... + c_n\Psi_n$, can be determined by solving task on definition of the eigenvalue of the risk operator $$\hat{R}\Psi = N\Psi,$$ where, $\hat{R}$ - risk operator. Portfolio with the minimum risk corresponds to the minimum eigenvalue N_{min}.
15	No analogue	The wave function described by the portfolio with the minimum risk is defined by components of eigenvector that refers to the minimum eigenvalue N_{min} or risk operator $\hat{R}$.
16	No analogue	Probability of the i-securities in the portfolio with the minimum risk is determined as the square of the i-component of the eigenvector of the risk operator that refers to its minimum eigenvalue.

INTRODUCTION

The successful development of any branch of human knowledge depends on how far it is possible to develop and introduce advanced achievements of the mathematical science. Undoubtedly, the leading position here belongs to the theoretical physics. Modern physical theories, which are excellently describing many secrets of macro and microphysics, for the whole decades have overtaken other sciences. To effectively study physical theories, it is already not enough to have an ordinary higher education. The long-term laborious studies of the scientific literature, research in the field of theory and practice are necessary for this. Only by the age of 35-40 years for a physics - theorist and by the age of 45-50 years for an economist, it is possible to reach the foreground of physical or economic sciences. Current scientific works in the quantum field theory, physics of elementary particles suggest to outsiders no more clear information, than the ancient Egypt hieroglyphs.

Scientific ideas, which have successfully built up a reputation for them in one science, are frequent applied with success in others. Physics is the leading donor of mathematical models. So, for example, a currently widely applied in economics the mathematical model of the variational principle was developed for problem solving in the theoretical mechanics in the second half of the XIX-th century.

By the beginning of the XX-th century, the physical science has faced a series of difficulties in its development. Therefore, a number of physical phenomena, in particular, in the microphysics, that were hard to describe by the easily explained concepts. In the 20^{th}, efforts of Luis d' Broille, Pual Dirac, Wolfgang Pauli, Max Rod, Erving Shredinger, Albert Einstein led to creation of the quantum mechanics that was rather effectively describing complicated phenomena of the microphysics with the help of concepts, which did not have analogs in that macro nature that is surrounding us.

Development of economic sciences, undoubtedly, was influenced by computers. On the one hand, they have facilitated and sped up solution of economic tasks. The speed of their solution has increased in thousand and more times. However, it became an obstacle to development of economic sciences as, mainly, not the ideology of problem solving was improved, but faster algorithms of solutions were created. Whereas the system of social production has been developing quickly and the economic phenomena became increasingly complex, mathematical approaches based on the old ideology, despite the power and speed of modern computers, could not solve the set tasks. As an example, the world economic crisis of the late 90th that should not be in the modern world, if we take into account the development of modern economic sciences and existence of a number of international economic institutes, which called to regulate world economic processes.

SECTION I. CONSUMER CHOICE AND PROBABILITY

Quantum mechanics has been successfully building up a reputation for itself already for more than seven decades as powerful and high-performance knowledge tool of a microcosm, and also that it describes phenomena that are similar to economic, for example, in the quantum mechanics, the measured value is compared with its most probable value. Besides there is an analogy: economic phenomena consist of interaction of millions of economic subjects, each of which operates in a definite economical and legal field, and phenomena of the microcosm take place as a result of interaction of multiple particles, which do not move arbitrarily, but follow certain laws of physics.

The purpose of this work is to explain mathematical apparatus of calculus of variations with reference to the description of economic phenomena, which we shall name the variational economic theory.

About a century has already passed since the times of Marshall and Hicks, and for this period the economic life all over the world has essentially changed. Now we have a huge set of the various goods with identical consumer properties, which try to find their buyer. In their attempt to increase sales of products, firms resort to various advertising campaigns, and as a result, sometimes it turns possible "to sell a comb to a bald headed". Customers are subject to a massive advertising attack. On our way to the office, we see various publicity boards, marvelously designed shop windows, newspapers and journals dazzling with advertising. We receive the portion of advertising information when we listen to our favorite radio stations or watch a TV set, at last, many of us discover, from time to time, in our mail boxes various advertising booklets. When purchasing new equipment, large enterprises carefully study various samples of equipment of various producers, and after careful estimation, we make a decision. Special departments and skilled employees are engaged in this problem, experts are attracted, and all this may take a rather long time. A small consumer, as a rule, has no possibility to use an expert, for example, when he buys a toothpaste, a detergent powder or a mineral water, and he defines their consumer values based on his personal experience of the use of these items, advices of his friends and influence of advertising. Basing on the fact that a skillful advertising campaign may essentially increase the sales of that product, the latter factor may play an important part in choice of an item of the product. A customer makes a decision on a purchase spontaneously, and the price is not always a deciding factor of such a choice. Having a limited budget, the customer tends to use his means to achieve maximum advantage. He may refuse buying the goods A in favor of the goods B if he will consider that the new portion of the goods B will be more useful to him, than a portion of the goods A. However, taking into account that the customer is under the influence of advertising, his actions may vary depending on the influence of various exterior factors.

In his theoretical researches, Alfred Marshall takes a certain consumer and considers his behavior while he makes decisions on a purchase of certain goods based on the so-called utility function. Although the utility function, undoubtedly, is a rather convenient source of formalization of many economic tasks, nevertheless, assumption about the existence of such a function is a certain hypothesis, the justice of which is disputable. On the other hand, it is natural that any person may make choices between any two sets of goods.

This methodological approach, however, it is not void of the following drawbacks: the consumer's preferences and tastes chosen at random are, generally speaking, the time-dependent values. We do not mention the fact that it is difficult to select two subjects with identical utility functions. In our opinion, the subject of the study should be a certain average statistical customer with average statistical views and needs, and the probability of purchase should be use as a measure of goods utility. The more consumer qualities are attributed to the goods, the higher probability of its purchase by a customer and its utility.

Let's construct a physical pattern that is adequate to the given economic pattern.

Consumer dispositions of the subject shall be characterized by us as a certain radius-vector $\bar{r} = \bar{r}(r_1(t), r_2(t), ..., r_n(t))$, where r_i corresponds to the amount of the bought goods $r_i = p_i x_i$, in other words, to the product of the price of the goods to their amount. The customer is primarily interested in the total amount of the purchased goods and the spent money, and not the cost of each portion. For example, when a housewife purchases 4 packs of some detergent powder, 1.5 kg each, at the price of 1.5 $ per pack or 3 packs of 2 kg each at the price of 2 $ per pack, the utility of both cases is identical.

$$1,5 * 4 = 2 * 3.$$

Therefore, while constructing the utility function, we will not be interested in the amount of goods and the price per unit.

Functional dependence $\bar{r} = \bar{r}(t)$ is stipulated by the fact that consumer preferences of the average customer are subject to time fluctuations.

According to the classical theory of utility, the customer spends his means so that to maximize the utility function, and cost of all purchased goods is equal to the capital of the customer r_{cl} = capital. In the authors' opinion, the customer's capital corresponds only to the most probable sum of the purchases made by him (fig. 1).

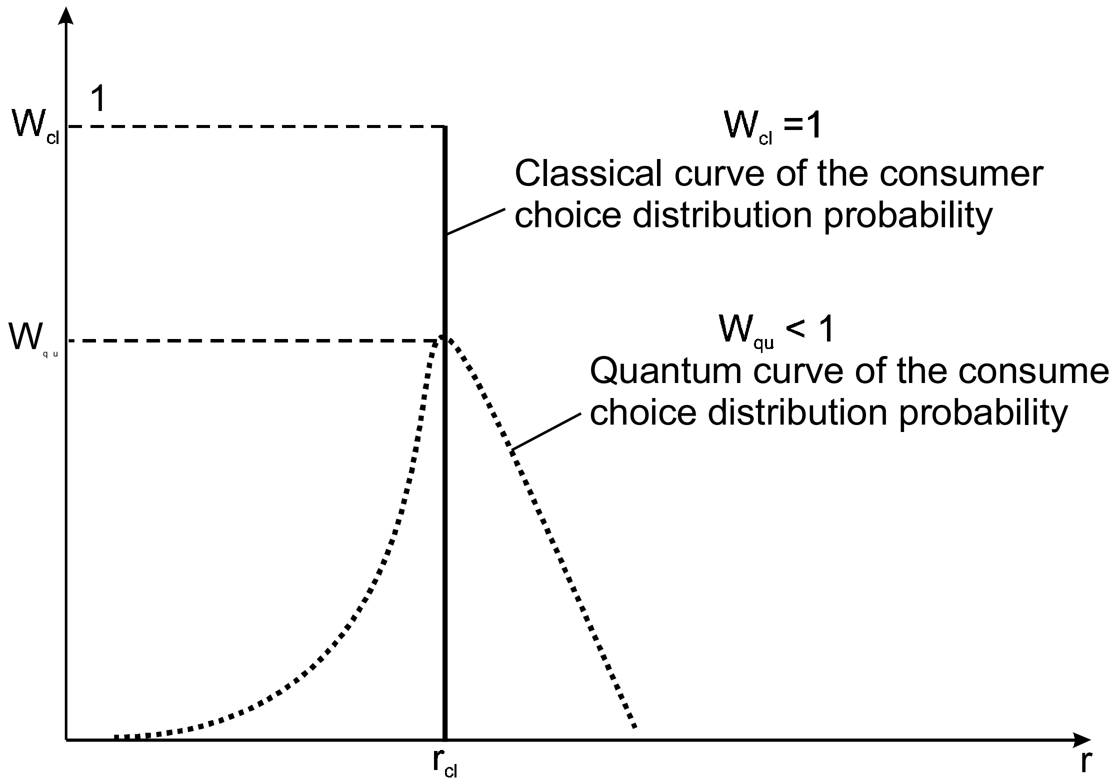


Fig. 1. Consumer choice distribution probability

According to Marshall, on the one hand, there is a customer with a certain money income, on the other, the market of consumer goods with already set prices. There is a question: how the customer will distribute his expenditures between various goods? For convenience of explanation, it is supposed that goods may be divided into very small-sized units. It is further presupposed that the customer receives a certain amount of utility from the purchased goods (thus, the total value of utility is the function depending on the amount of the purchased goods), and he will spend his income in such a way that he will receive the greatest possible amount of utility. However, utility will be maximized on conditions that limiting unit of each type of expenditures guarantees the identical increase of utility. You see, in this case, when we change the direction of means expenditure to various goods, the utility loss for goods, expenditures on purchase of which were reduced, will exceed the benefit of utility of goods, expenditures for purchase of which have been increased (if we proceed from the principle of the decreasing marginal utility). Therefore, no matter how the direction of means expenditure was changed, the total utility should decrease. As, under condition of available small-sized commodity units, the difference between two succeeding utilities may be neglected, and the received conclusion can be formulated by us as follows: marginal utilities of various purchased goods should be proportional to their prices.

Thus, in his reasoning, Marshall is based on the condition of maximization of the total utility and by basing on the law of the decreasing marginal utility, concludes that the marginal utility of goods should be proportional to their prices.

Marshall began his research from the theory of utility, and Pareto (without reference to other researchers) repeated his actions. However, Pareto at once has paid attention to the problem of interconnected - complementary and competing - goods, instead of the relation between the curve of the decreasing marginal utility and the demand curve (as it was done by Marshall).

During his research of the interconnected goods, Pareto started to expand the former views of this problem and eventually has produced a revolution of these views.

We suggest the modified graphical method of Adjourte for study of the problem of interconnected goods (namely, review of curves of equal probability).

Considering only one item of goods, we may build a curve of the aggregate probability of the purchase. For this purpose, we shall lay the mathematical expectation of the amount of these goods on one axis, and on the other – the values of the aggregated probability of the purchase of the given amount of goods. Similarly, when we are interested in two goods, we may build a area of equal probability. Laying the probabilities of the purchase of the two goods, X and Y, on two horizontal axes, we shall receive the graph, any point P of which will have a certain set of defined amounts (PM and PN) of our two goods. From any of these points, we may, by passing over to the three-dimensional image, lay off the ordinate (its length is the value of probability of their purchase, ensured by current set of these two goods). By connecting tops of the ordinates, we shall receive “the area of equal probability of the purchase – sale” (fig. 2).

For simplification of the research, instead of the three-dimensional model, it is possible to resort to the image on the plane (fig. 3). Similarly, by laying off expectations of the amount of two goods X and Y on two axes, we plot the projection of the area of equal probability (a dashed line on fig. 3). These are the curves of equal probability. Numbers of equal probability curves were taken arbitrarily: for convenience in research, we started their indexing from 1 and produced ascending order.

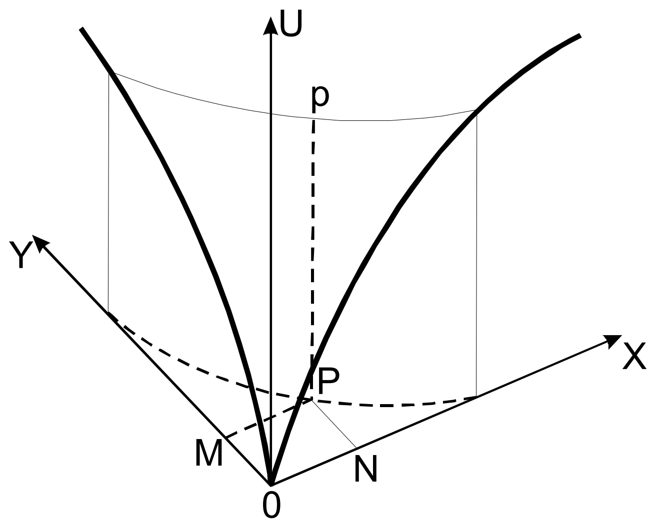


Fig. 2. Area of equal probability

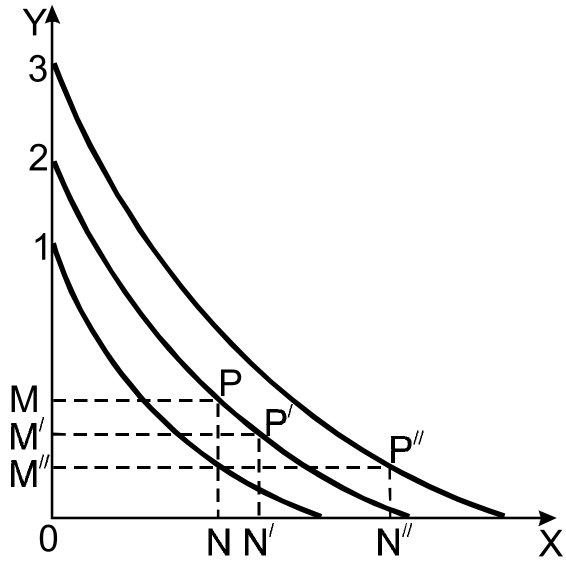


Fig. 3. Curves of equal probability

Curves of equal probability connect all points of identical height in the third dimension, in other words, of identical probability. If points P and P′ are on one and the same curve of equal probability, the total probability of the purchase for the owner of the set of goods PM and PN is equal to total probability of the purchase for the owner of the set of goods P′M′ and P′N′. If P″ is located on a higher curve of equal probability compared with P′ (curves should be enumerated to distinguish the high ones from the low ones), than the sets P″M″ and P″N″ will produce higher total probability, if compared with the sets PM and PN.

What will be the form of these curves of equal probability? Curves of equal probability should be sloping to the right, as the limiting probability of the purchase of each goods is positive. If the limiting probability of the purchase of the X goods is expressed by a positive value, then the increase in the number of these goods that is not accompanied by any change of the number of that Y goods will point to the growth of probability of demand for the given goods (therefore, it would bring us to a higher curve of equal probability). Similarly, and a simple shift upwards along the graph should bring us on a higher curve of equal probability. The customer may "stay" on the former curve of equal probability only when the change in the number of goods is compensated: the amount of the X goods grows and the amount of Y goods drops or, on the contrary (it means that curves should slope to the right).

The significance of the curve slope passing through some point P, actually is rather definite and important: it represents the amount of Y goods that is necessary for compensation for the loss of the least unit of X goods to the average individual. Thus:

Increase in probability of purchasing of a certain number of Y goods =
= Number of the purchased Y goods x Limiting probability of Y goods purchasing,

Decrease in probability of purchasing of certain of X goods =
= Number of the lost X goods x Limiting probability of X goods
(on condition that these values are small).

Then, (as the benefit and the loss are equal) the curve slope may be presented as follows:

$$\frac{\textit{Number of the purchased Y goods purchasing,}}{\textit{Number of the lost X goods purchasing}} = \frac{\textit{Limiting probability of X goods}}{\textit{Limiting probability of Y goods.}}$$

The slope of the curve passing through the point P shows the relationship between the limiting probability of purchasing of the

X goods and that limiting probability of purchasing of Y goods in the case, when the individual has the PM number of X goods and PN number of Y goods.

Let's return to the form of the considered curves. It seems that there should exist some way of graphical representation of the decrease in limiting probability of purchasing. At first sight, it may seem that it is possible. Moving along the curve of equal probability, we get more and more X goods and less and less Y goods. Growth in the number of X goods reduces its limiting probability of purchasing and drops in the number Y goods - to the increase of its limiting probability of purchasing. And due to both reasons, therefore, the curve slope should decrease. Falling curves, whose slopes decrease following our "movement' to the right, will be convex, as it is depicted in the figure (see fig. 2).

Now we shall consider a rather interesting property of the curve of equal probability, which has routed the development of the given theory to another channel (in comparison with the theory of Marshall) and has allowed receiving results of the high theoretical importance.

Let us presume that the average customer with a certain money income spends all his income for purchasing of two goods - X and Y. Let us presume that expectations of market prices for these goods are given. Then, if we would like to know, how many goods the customer purchases, we should address directly to the assemblage of curves of equal probability belonging to him, even if we know nothing about the value of probability of his purchasing these goods. Let us lay off the OL curve segment on the axes X (fig. 4): it indicates the number of X goods, which the customer might get if he spends all his income on this goods; let us lay off the OM curve segment on the axes Y: it indicates the number of Y good. Now we shall connect points L and M. Then, any point of the right line LM will correspond to the defined set of two goods, which the customer might get if he spends his income. If the customer is "moving" along on LM from point L, then, for purchasing of a certain number of Y goods, he should refuse from purchasing of the defined number of X goods, which is dependent on relationship between price expectations of these two goods, and latter is defined by the slope of the LM right line angle.

The curve of equal probability may pass through any point of the right line LM, but, as a rule, LM will intersect it. At the same time, the cross point cannot be one of the equilibrium points. By moving along the right line LM in both directions, the average customer can always hit the higher curve of equal probability. In other words, at this point the probability of purchasing for the customer is not maximal.

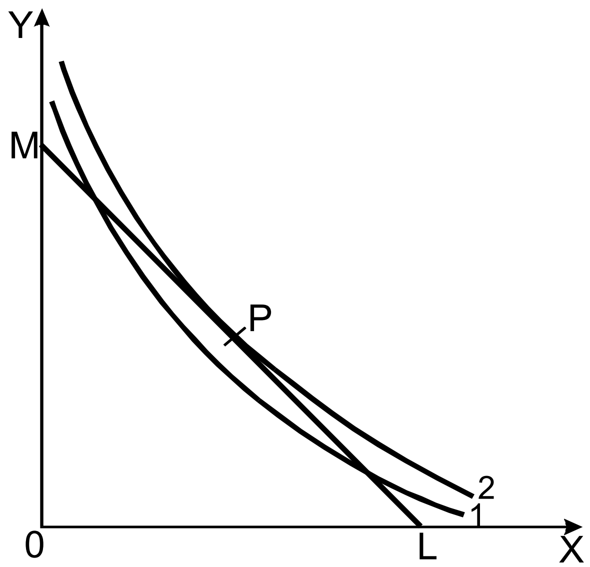


Fig. 4. Equilibrium price

Only in case of LM touching the curve of equal probability, the probability of purchasing will be maximal. From the point of contact, the customer may get only on the curve of equal probability located below, independent of direction of movement along the right line.

Contact of the price line and the curve of equal probability expresses the proportionality of values of limiting probability of purchasing and the price.

Now, it is possible to explain the theory of limiting probability of purchasing using the "language" of the curve of equal probability. We have received the necessary result by rejecting some initial prerequisites.

Therefore, to determine the quantity of goods purchased by the individual under the given prices, it is necessary to know his area of purchasing probability. The traditional model of Pareto offers to analyze only the set of curves of equal probability, that (in comparison with the area of equal probability) carries much less information. It informs that the average individual prefers one concrete set of goods to the other, and **by what degree** the first set is more preferable than the second - this is the question without an answer. It means that to get the verified data for our research it is necessary to consider the area of equal probability.

Nowadays, the major part of the EEC countries' population is above the level of ordinary physiological needs: the customer frequently changes the set of the purchased goods under the influence of fashion and advertising.

The higher the probability of purchase of the goods, the higher its price, and it means, that the lower the probability of purchase of the goods, the lower its price. If the seller attempts to sell the goods at a higher rate than the most probable price, he would require much more time for this comparing with the most probable price of purchase. In his turn, the seller may influence the customer by way of advertising and thus increasing the wish to purchase goods for the higher price. While buying the goods, the customer pays attention to its appearance, on the prestige of its possession, and his actions are influenced by his habits, fashion and other subjective factors. This phenomenon has turned possible with competition. The larger the diversity of the same type of goods, the greater the influence of the non-price factor on his purchase. Not being the highly competent expert in production of the given goods, the customer is not capable of unbiased evaluation of the goods quality. It is hard to evaluate, what kind of the mineral water or a soft drink is better to satisfy the thirst. The final judgment on purchase or refusal to purchase the given goods, in many respects, depends on its packing, where and how these goods are sold, and on the skill level of the serving staff (i.e. on services). The buyer is constantly subject to the massive advertising attack on the part of mass media, and therefore, distribution of his limited budget depends on the influence of this or that product advertising.

That is why, to tell that the equilibrium market price is defined by the buyers' desire to pay a certain price for the goods, and the sellers - to sell it for the given price, will be too simple and incorrect. It would be more correct, in our opinion, to say the following about the definition of the equilibrium market price: **the equilibrium market price is determined by the momentary, instant desire of the buyer to acquire the given product for the most probable price, and the seller to sell it for the most probable price he is willing sell for.**

In the system of free entrepreneur activity, anybody is specially engaged in finding solution of three economic problems: What? How? For whom? We shall give you an economic example, with reference to London. Without a continuous flow of goods moving into the city and from the city, it, within one week, would be on the verge of famine. This is the flow of the diversified foodstuffs necessary for its inhabitants. From the most remote corners of the country the goods are traveling, for days and sometimes even for months to get to the capital. How can the population of such a big city sleep quiet at night and not feel the mortal fear that these complicated economic ties, the existence of the whole city is depending on, one fine day may be disrupted? All of them exist without any enforcement or centralized management on the part of any competent institution!

Only the fact of operation of the competitive system of markets and prices is the convincing proof of fact that this system with all its virtues and shortages is everything, but only not a system of chaos and anarchy. It follows a certain inner order and is subject to certain regularities, it operates. The competitive system is a complicated mechanism of involuntary coordination acting through the system of prices and markets; this is a mechanism of links that serves for joining knowledge and operations of millions of various individuals. Without the help of experts, this system solves one of the most complicated problems that only can be imagined: the problem that includes thousands of unknown variables and relations. Moreover, what the most remarkable is that nobody has invented this system. It has been simply developing and, is changing similarly to a human nature. At the same time, it satisfies at least the first requirement presented to any social organism: it is capable to survive and it survives!

Adam Smith was shocked after he had discovered a definite order in this economic system and offered the principle of the "invisible hand": each individual pursuing only his own interests is as if routed by some invisible hand for everybody's benefit. Only those goods are manufactured and sold that are most likely to have the greatest probability of being sold. Moreover, the greatest probability of being purchased belongs to the goods that are consumed by the community. Therefore, any interference of the state in to the sphere of free competition will most likely result in negative consequences. Smith recognized the known narrowness of his conclusion, but only later, economists have discovered this regularity: benefits attributed to the free entrepreneurship are fully inherent in it only when there are all constraining and balancing factors of the perfect competition. The perfect competition is the case, when one subject of the market itself has a minor probability to influence the market price. If he owns enough grain, goods or labor that will allow him to influence the "most probable market price", as well as with a high degree of probability to influence the fall or rise in prices, then the concept of "perfect competition" is substituted by a defined degree of monopolistical imperfection and the virtues of the "invisible hand" are being diminished.

How operates this involuntary automatic mechanism of price formation? The simple description of the competitive system of profits and losses is quite possible (even various aspects of human labor have their prices, the so called "wage rates").

Everybody receives money for the sold product and can buy whatever he needs for this money. If probability to sell any goods is increasing, then probability of its production is also increasing. Growing demand in these goods entails the increase of its purchase probability even at the increased price. Similarly: if some goods in the market are in abundance, the probability of its purchasing at the last market price is lower.

All the above reasons are also true in relation to the market of labour, land and capital. If the want in drivers is higher than in doctors, probability of finding job for a driver is higher. The price of drivers' labour, their hourly pay will tend to grow while the wages of doctors will remain the same or even reduce. Under other equal condition, the probability of doctors retraining for drivers increases and the probability of drivers receiving the diploma of a doctor decreases. Similarly, this or that hectare of land, with the greater degree of probability, will be sown by corn, instead of wheat, if customers will pay more for it.

In other words, we have a ramified system of checks and errors, successive approximations to the system of prices and production equilibrium.

Contraposition of the demand and supply, prices and costs simultaneously helps to solve the three below mentioned problems.

1. What exactly will be produced? This it is determined by customers: the more persons wishing to buy goods, the higher probability of its purchase and the price, at which it will be sold.

2. How things are made and brought into life? The answer to this problem is determined by competition among various producers: to produce the goods which are meeting all consumer properties does not mean to sell it.

In our opinion, it would be logical to limit the goods production time by the period from purchase of raw material to the output of the finished product from the conveyor. This process goes on during sale of the goods. Conditions of the goods sale endue it with new properties in the imagination of potential customers. An important part is played by packing of goods, payment conditions, after-sale guarantee and maintenance. All this is associated with the goods and influences the buyer's choice of this or that trademark.

Industrial method, which is, owing to its higher performance (both by the physical size of production and cost), at present, the cheapest, displaces a more expensive method. However, this is not always true. If there are goods - competitors (for example, when selling fashionable clothes, cosmetics, cigarettes), the seller should invest big money in advertising of goods to be sold. Bright colorful packing, modern design lead to higher production costs, and though the consumer properties of the goods do not change, their price grows. Nevertheless, in this case, the probability of the purchase of the goods is increasing, as the non-price factors render an essential influence on the customer.

3. For whom are products manufactured? This is determined by demand and supply in the market of productive services: by wages, land rent, percents and profit, which increase the income of each person (expressed in monetary units), in the definite relation with the income of other persons and with the income of the community. The way the final distribution of the income is carried out depends on the initial distribution of property, on acquired or inherited rights.

Not only the customers' choice determines, what goods should be produced: alongside with the customers' choice there exist production costs and the ability of producers to offer the goods. Like the broker, who may help to conclude the transaction between the buyer and the seller, the auctioneer in the goods market operates as an intermediary, which conciliates the opposition of the customers' demand and the producers' supply. In the same way, we may consider the auctioneer, who presides over the market of the skilled labour and as the intermediary conciliates the business labour demands with the ability of the community to offer it. The authorized representative of community, who determines how to produce the goods, is the one, who tries to receive profit and to decrease the factorial production costs of each item of goods, and also the one, who is mercilessly penalized by competition if fails to take advantage of the best methods of business.

The drawback of the described pricing system is that actually competition does not approach the “perfect” one. Firms do not know the probability of change of customers’ tastes; therefore, they may cause overproduction in one sphere and underproduction in another. By the time they learn about it, based on their own experience, the position may change again. Besides, in the competitive system, one producer does not know, what methods are used by others, and therefore, production costs cannot be reduced to a minimum. Save to competition and by keeping secret the acquired knowledge, it is possible to achieve the same success, as well as to maintain the high level of production.

More serious deviations from the “perfect” competition are caused by monopolistic elements. An “imperfect competitor” is anyone, who purchases or sells any goods in quantities that can, with the greater degree of probability, influence the price of these goods.

All economic life is an interlacing of elements of competition and monopoly. The dominant form is the imperfect (monopolistic) competition. This is a statement of facts, not a moral condemnation. The maximum that can be achieved is more or less significant approximation to the perfect competition.

A producer cannot set the price only of his own desire and receive profit. He should take into account the prices for similar goods of rival firms. Even if he makes a detergent powder that possesses unique properties and has a well-known trademark, all the same, he should take into account prices for the powder of other firms and other washing products.

All people, be they directors, workers or farmers, have a dual attitude towards competition: a positively and a negatively one. It is pleasant to everything when it allows expanding the market, but it can be called “dishonest”, when the other side of competition faces you. Worker, whose existence depends on evaluation of his labour in the market, may the first to become indignant, if competition threatens him by lower wages. Agricultural firms are continuously pressing the state with the purpose of limiting production and thus, rising prices.

Some of the major factors, which lead to monopolistic management of business, are likely to be inherent in large production. Especially, this is manifested in the dynamic world of technical innovations.

Competition of the uncountable producers would have been simply ineffective in many fields and could not exist too long. Trademarks, patents and advertising frequently lead to other imperfections of the market. Therefore, it is impossible to create "perfect competition" under the law. The task is in attaining the pattern of "perfect competition", in other words, acceptably effective "feasible competition".

Let's begin with the demand. Everyone might be convinced that the number of things purchased by people always depends on their price: the higher the price of the goods, the less are purchases, and the lower its market price, the more (considering other equal conditions) items of such goods will be purchased.

Hypothetical table 1 represents the example of the demand chart. At any price, for example 100 dollars per one ton, all customers will purchase in the market, a certain amount of wheat - in this case, 9 million tones. At a lower price, say, 95 dollars per one ton, the amount of the purchased wheat will increase and will reach 10 million tones. According to the data of table 1, comparing column 3 and 2, we may determine the possible purchased amount at any price.

Table 1. Relation between the wheat price and the volume of demand

	Prices in US dollars per one ton	Demand in million tons per one year
A	100	9
B	95	10
C	90	12
D	85	15
E	80	20

Figures of table 1 can be represented graphically. On fig. 1, along the axis of ordinates we lay off a variable line of various prices for wheat in US dollars per one ton. Along the axis of abscissas, we shall lay off the amount of wheat (in tons) that will be purchased per one year. The curve is sloping from the northwest to the southeast. This important feature has its name: the law of gradual decrease of demand. Its reasons are easy to determine. When the price of wheat is rising up to the sky, it can be purchased only by the rich, and the poor have to do with the rye bread. If the price of wheat is still high, but not to such a degree any more as before, it may be bought, in small amounts, by persons with moderate means, who are also the big lovers of white bread. Thus, the first important reason for justifying the law of gradual decrease of demand is that the drop in pricing entails extension of demand, i.e. attracts new buyers. Somewhat less obvious, but still important the second reason in favour of this law, namely: each drop in pricing increases the probability of additional purchases by each consumer of the given goods. And, on the contrary, higher prices may force any of us to purchase less. Why, when the price of the goods rises its amount tends to decrease? It is explained by two main reasons. If the price of any goods is increasing, the buyer, with the greater degree of probability, will purchase other goods instead of them. Besides, when prices grow, he turns poorer and is now in possession of a smaller real income. Then, certainly, the buyer starts to consume the whole set of the goods in a smaller amount.

At the same time, when more goods are available in the market and the price goes down, the customer increases the purchases of these goods. We shall give you an example. If clothes are very expensive, the buyer's demand is limited only by the amount that is necessary for his every day life. Probability of purchase of clothes to satisfy other needs is low. At reduction of prices on clothes, the buyer starts to break the uniformity of his wardrobe: he purchases on two pairs seasonal footwear (for example, of different color or different style shoe, boots and etc.), buys clothes for other purposes: for parties, sports, for his new image, etc. At further cuts in prices, the probability of new things purchase of new clothes will increase.

We have described the demand, now we shall pay our attention to sellers. Let's dwell on the supply chart. Unlike the demand curve, the supply curve will usually go up to the right, from the southwest to the northeast. At high prices of wheat, farmers use plots for other crops to cultivate wheat. Besides, now the farmer may purchase more fertilizers, employ additional workers, use more machines. He may even dare to cultivate wheat on less fertile plots. It means that at higher prices there is a tendency of increase of product output.

To determine the process of the competitive market price formation, it is necessary consider the demand analysis and the supply analysis. Up to what level the price may actually change? How many products will be produced and consumed in this case? Let's presume that sellers have a store of 100 million tons of grain. We shall combine data of the supple and demand tables into one table (table 3) and calculate probabilities of sales and purchases of grain at the different prices.

Table 2. Relation between wheat prices and volumes of supply

	Possible prices in US dollars per one ton	Wheat supplied by sellers in million tons
A	100	18
B	95	16
C	90	12
D	85	7
E	80	0

Table 3. Probability of grain sale and purchase

	Possible prices in US dollars per one ton	Demand in million tons	Volume of supply in million tons	Pressure on prices	Probability of purchase	Probability of sale
A	100	9	18	Down	0.09	0.18
B	95	10	16	Down	0.1	0.16
C	**90**	**12**	**12**	**0**	**0.12**	**0.12**
D	85	15	7	Up	0.15	0.07
E	80	20	0	Up	0.2	0

According to the classical theory, the only balancing price (in other words, the only price that can be maintained) is such a price, at which the sum of the supply and the sum of the demand are equal; **the competitive equilibrium is always at the cross point of curves of demand and supply.**

At first sight, everything is simple. However, we shall consider another example. Let's presume that a certain producer makes macaronis. If they are packed in ordinary-looking packing, then, due to abundance of the competing goods, there will be not many buyers. However, as soon as the product is packed into the beautiful polypropylene packing, the number of persons interested in purchasing this product will start to increase. Why? As many buyers will associate the more attractive products with higher quality products (and all this despite of the fact that the contents of this pack have not changed and the consumer has to pay production costs of new packing, i.e. despite the rise in price of the goods).

Or one more example: the packed office paper may be sold at different prices depending on how many buyers know its trademark; if the same paper offered in another packing, the persons interested in its purchasing will be less and its price will be lower.

The price at which consumers purchase the goods is higher than the price at which these goods are sold by producers to trading companies. The seller's profit depends on how successful he will be in convincing the buyer to buy goods from him at the higher price compared with the one he has received from the producer. The rise in demand for the goods is stipulated by the rise of probability of its purchasing by the buyer, and it there may be two reasons: either the lower whole sake prices for the goods or the efforts of trading companies focused on their sales. As all advertising and sale expenses are eventually have to be paid by buyers, one can observe the situation, when the rise in prices may increased the product demand. A simple example: production of the firms Coca-Cola, more than a half of cost price of the sold soft drinks of which is advertising costs. The presence of this phenomenon is possible only in case of the limited access to information on the sold goods on the part of the potential buyers. Besides, the process of sale and purchase itself is unique, i.e. the same products may be sold at various prices to different buyers. A more prosperous person may pay for a book higher, as the marginal cost of money for him is lower, than of the one with less money. When buying automobiles or houses worth of tens of thousand dollars it is possible to bargain between the seller and the buyer about hundreds of dollars, and during the purchase of apples for some dollars – about tens of cents. In both cases, if the transaction was completed, the seller may be pleased, if he has reduced the price, in the first case, by 200 $, and in the second – by 0.20 $.

As the buyer is interested in not only the price of the product, but also in other factors (for example, in after-sale services), in our opinion, production of goods should subdivided into four stages (fig. 5).

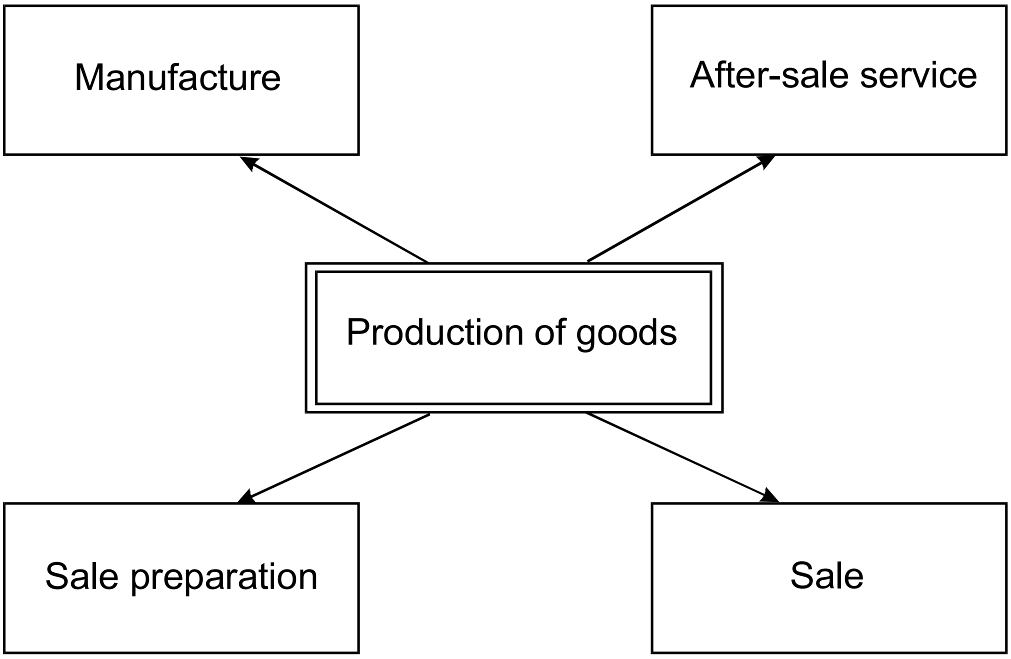


Fig. 5. Stages of the process of goods production

The rise in demand for the goods may actually mean the rise in demand for the goods sold in new packing. This it will not necessary stimulate its production, especially, if there are many goods - competitors (for example, yogurts, mayonnaises, ketchups). It would likely to stimulate its advertising. This fact is explained by the fact that in case available set of goods - competitors with identical or approximately identical consumer qualities, only whose ones will be sold that will require more efforts to be sold (this may be advertising, packing and after-sale services). The further increase in production of goods in the environment of raised demand for it occurs in the event, if the marginal utility of expenditures on sale of the commodity unit is lower, than limiting expenditures on production of this unit.

Let's do the same in other words, done by the auctioneer. In other words, we walk the path of trails and errors. Whether the position A in table 3, when the wheat is sold at 100 $ per one ton be maintained during this or that period? It is obviously no. Producers will deliver monthly 18 million tons at the price of 100 $, but an amount of wheat in consumers' demand will be equal to only 9 million tons annually. As their stocks starts to build up, the competing sellers of wheat will reduce its price a little. Thus, the price will tend to drop, but it will not decrease indefinitely and will not be lowered up to zero.

For better understanding of this problem, we shall address to position E, where the price per 1 ton is equal only to 80 $. Can this price be maintained? Obviously - no, as at this price, consumption will exceed its production. Warehouses will start to become empty and the dissatisfied consumers, which already cannot purchase the wheat, will offer higher price for it. Only in position C, at the price level of 90 $ per ton the level of the demand of 12 million tons annual will be exactly equal to volume of the supply.

Now we shall consider the situation, when the seller has conducted an advertising campaign for attraction of new buyers. This has induced new buyers to purchase his goods above the earlier set point of equilibrium. It seems that producers should increase production of this product, however, the profit received from attraction of new buyers has to be spent for advertising campaigns, printing materials and partially settles with dealers, and as a result, the producer has no objective incentives to increase production. That means that despite of the expanding demand for the goods, its production and supply will stay unchanged. Moreover, this position may turn rather stable.

Following the classical theory, the demand curve should slope upward and the cross point with the curve of total supply should be located a bit to the right. However, due to lack of the producer's incentives to increase production these curves will never meet. In this case, the equilibrium market price is not determined by intersection of the demand and supply curves.

To eliminate the above drawback of the theory, we shall give you our own definition of the market equilibrium using the language of the probability theory: **the equilibrium price is the price at which the probability of the purchase of the goods is equal to probability of its purchasing.**

SUMMARY TO SECTION I

1. Consumer tendencies of the subject we shall characterize by a certain radius-vector

$$\bar{r} = \bar{r}(r_1(t), r_2(t), \ldots, r_n(t)),$$

where r_i corresponds to the number of the purchased goods $r_i = P_i X_i$, in other words, the product of the price of the goods and its quantity.

2.
$$\frac{\textit{Number of the purchased Y goods purchasing,}}{\textit{Number of the lost X goods purchasing}} = \frac{\textit{Limiting probability of X goods}}{\textit{Limiting probability of Y goods.}}$$

3. The equilibrium market price is determined by momentary, instant desire of the buyer to buy the given goods at most probable price and the seller's consent to sell the goods at the most probable price.

4. The equilibrium price is not always determined by intersection of the demand and supply curves.

5. The equilibrium price is the price, at which the probability of the goods purchase is equal to the probability of their acquiring.

SECTION II. VARIATIONAL ECONOMIC THEORY

2.1. Theory of decision making under risk

The theory of decision-makings has three possible situations of choice of the decision:

1) Choice under determinacy - when the result of the decision is determined and may be determined beforehand;

2) Choice under risk - when the result is not verified exactly beforehand, but there is information on probability distributions of possible consequences;

3) Choice under uncertainty - when the result is accidental and there are completely no information on probabilities of decision consequences.

Despite of the precise classification from the practical point of view, distinction between the second and the third situation is rather fuzzy, if probability is understood as the so-called subjective probability (probability as the degree of a person's certainty in this or that result).

The economic theory presumes that the "*homo economicus"* (in other words, the person capable of rational decision making basing on the principle of the greatest benefit) always has conceptions that are based on these or those considerations, on that degree of riskiness of this or that alternative. These considerations are grounded on the degree of the individual's reliance (subjective probabilities) on the results of various consequences of the accepted decision.

Let's consider, as a result of the decision, some unique index (for example, the amount of income, profit, wealth and etc.) the actual value of which depends on the accepted decision and some random factors. It means that each variant of decision has it own probable distribution of the result. Then, the choice under risk is the choice among probable distributions appropriate to each variant of decisions.

For example, the decision is whether to purchase or to not purchase a lottery ticket (lottery is a very popular example in the literature on problems of risk). The ticket costs 5 dollars, and the probability of winning of one million dollars is equal 0.005 %. Alternatives are presented in table 4, and existing possibilities - in table 5.

In other words, each variant of the decision entails some distribution of the result probability.

Table 4. Alternative decision on purchasing the lottery ticket

I To buy	II Not to buy
Income = 0 with probability of 1	Income = 999 995 with probability of 0.0005 % Income = -5 with probability of 99.9995 %

Table 5. Probability of consequences of the accepted decision

Income	Distribution of the result consequences	
	To buy	Not to buy
-5 0 999 995	0 % 100 % 0 %	99.9995 % 0 % 0.005 %

We shall base on presumption that a person is always capable of making a conscious choice between the available alternatives of the decision - to choose the most preferable and probable distribution. We also think that these preferences are subject to some natural rules (i.e. they are rational).

If the admissible area of choice between the risk and the income is limited, the best possible variant of the decision will be the choice that insures the highest level of benefits (utility).

2.2. Prime model of choice of decisions and the premium for the risk

Let's presume that the risk can be measured by some way. It means, that each variant of the decision possesses:

- A value of the expected ("average") income;

- Some index that describes the degree of risk of the given decision (we introduce the square of profitability rate of change).

In this case, the person's preferences may be represented with the help of the standard set of microeconomic curves of equal probability or indifference curves.

Each point on the coordinate plane of the figure 6 defines some level of the expected income and an appropriate degree of risk of its getting. The points laying on the same indifference curve show combinations of the degree of risk and the expected income, which are equivalent for this person.

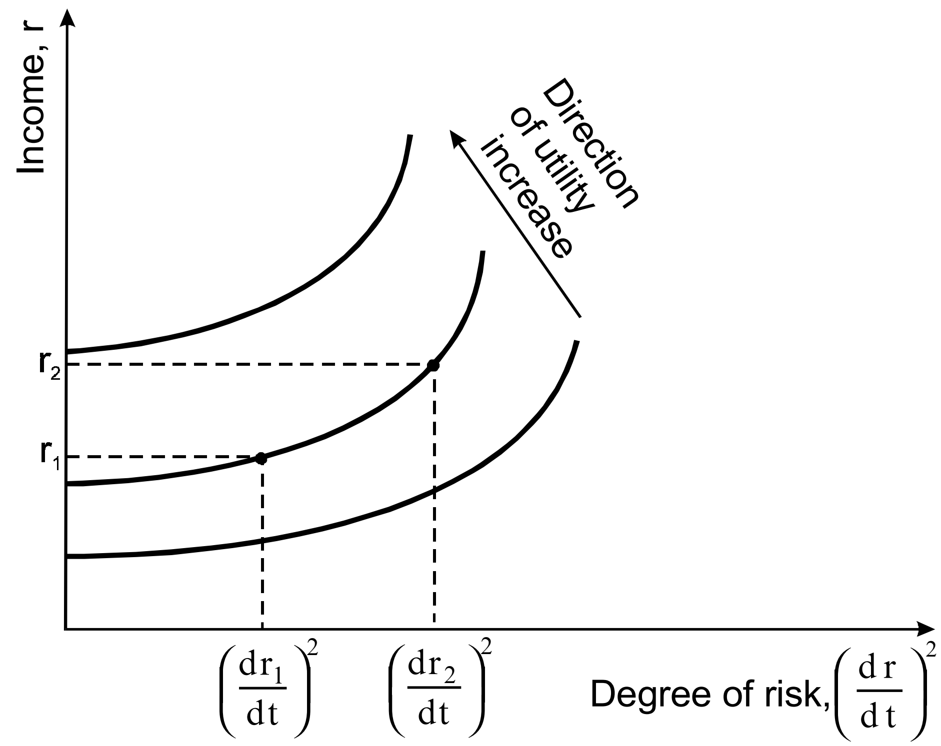


Fig. 6. Dependence between risk and income

If the increase of income (fig.6) from r_1 to r_2 is accompanied by increase of the degree of risk (that is determined by the square of profitability rate of change) from $\left(\frac{dr_1}{dt}\right)^2$ to $\left(\frac{dr_2}{dt}\right)^2$, then, both alternatives are equivalent for that person. The form of indifference curves in figure 6 corresponds to the supposition that the expected income, if we follow the terminology of the theory of consumer choice, is the "positive" asset (the more the income, the better), and the risk is the "anti-asset" (the increase of the degree of risk worsens the position of an individual).

The increase of the expected income, for example by the value $(r_2 - r_1)$ that compensates to the person the loss of benefits caused by the rise of risk (in this case from $\left(\frac{dr_1}{dt}\right)^2$ to $\left(\frac{dr_2}{dt}\right)^2$, we shall call the premium for the risk.

Slope (to be more exact, the slope ratio) of the indifference curve is the limit rate of replacement between the risk and the income (we shall designate it as MRS_{rr}) that indicates how should increase the income (the premium for the risk) to compensate the increase of the degree of the risk to one unit:

$$MRS_{rr} = \Delta Income / \Delta Risk.$$

In other words, the limit rate of replacement characterizes the relative value of the risk and the income for the individual at the given level of the former and latter (in other words, the individual premium for the risk).

To complete the model we shall presume that the choice for the subject is limited by the environment (say, the market). Let's presume, for example, that there is some feasible set of combinations of the risk and the income, designated by the shaded area in figure 7.

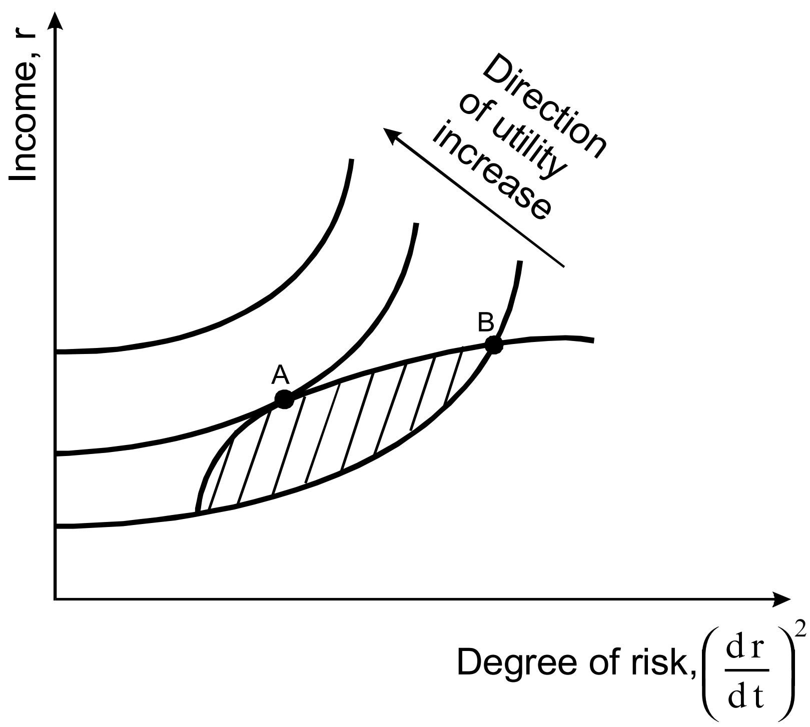


Fig. 7. Optimal choice under risk

Basing on the principle of rationality of choice, the person will prefer that alternative from several possible that will provide to him with the best standard of living. In figure 7, this is the point A, appropriate to the best possible one from all indifference curves.

We may say that the slope of that boundary of the allowable area represents the market premium for the risk - the increase of income that can be received, if the degree of risk is increased by one unit. Whereas the best choice is the point, where the highest indifference curve between the risk and the income touches the boundary of the allowable area, i.e. the point, where the slopes of both curves are equal between themselves.

The condition of the best choice is will be the equality:
The individual premium for the risk = the market premium for the risk.

2.3. Disinclination to risk

The form of indifference curves that is similar to the ones depicted on figures 6 and 7 describes the person's preference, which is not inclined to risk. The individual is not inclined to risk if the rise of the degree of risk at the constant expected income has a negative effect on his benefits. In other words, the risk for the person, who is not inclined to risk is always an "anti-asset".

People differ by their attitude toward risk. A person is inclined to risk, if the increase of the degree of risk rises his benefits, and is neutral in his attitude toward risk, if change of the degree of risk has no effect on his standard of living. The indifference curves, which reflect the properties of inclination and neutrality toward risk, are given in figure 8.

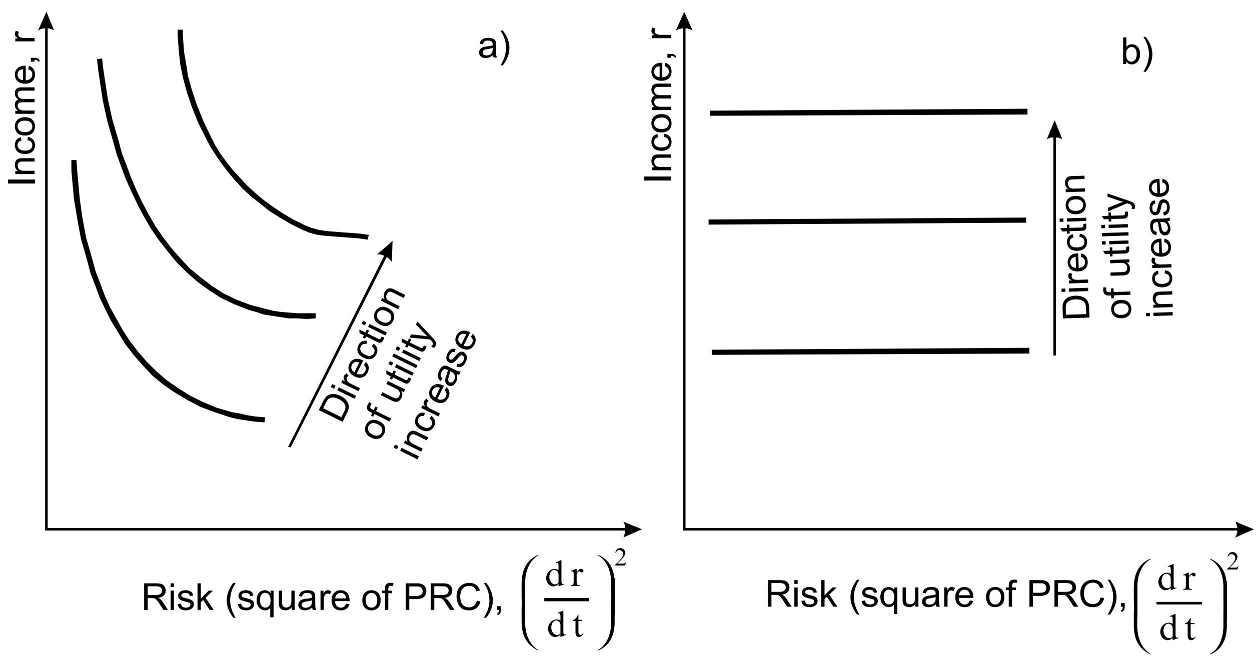


Fig. 8. Indifference curves of the person, who is inclined to risk (a) and is neutral in his attitude toward risk (b)

Let's point out the important behaviour singularities of people with the different attitude toward risk. For the person, who is not inclined to risk, the drop in the degree of risk, even at the constant expected income, increases the attractiveness of the decision, whereas in case of neutrality, only the expected income is important. For the person inclined to risk a more risky decisions is always more preferable.

The economic theory presumes that the absolute majority of people are not inclined to risk. However, the degree of such disinclination may differ. Indifference curves in figure 9 a characterize the person, who is relatively less inclined to risk in comparison with the subject, the indifference curves of which are given in figure 9. The greater the increase of the income that is necessary to compensate to the person the rise of risk, the less he is inclined to risk. Referring to investments, a higher disinclination to risk in the financial slang is called "conservative" or the cautious approach. Accordingly, the lower the disinclination to risk, the more "aggressive" (risky) is the behaviour of the investor. Formally, within the framework of the considered model, the degree of disinclination to risk is characterized by the value of the limiting rate of replacement between the risk and the income MRS_{rr}: the higher is this value the more the person is not inclined to risk.

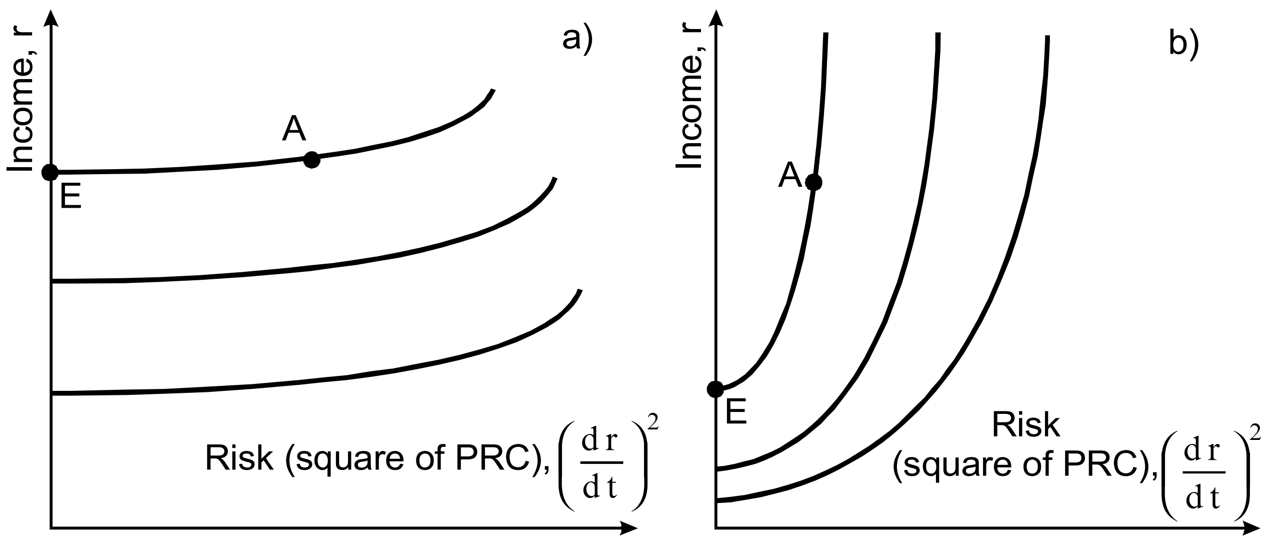


Fig. 9. Differences in the degree of inclination towards risk:
a) "aggressive" investor, b) "conservative" investor

The degree of disinclination towards risk can be described in a different way. The value of income in absence of risk, which is equivalent (lays on the same indifference curve) to certain risky alternative A and is called as the determined equivalent of the decision A. The more the disinclination to risk, the less the determined equivalent of one and the same decision. In figure 9, the determined equivalent of the decision A is represented by the point E.

Attitude towards risk and utility of wealth

The model of the choice of decisions above is based on the supposition of possible measurement of the degree of risk connected with this or that decision. This assumption is very strong. We may reject it by offering a more common criterion of choice of decisions.

The most spread supposition in economic theories is that the person, who makes the choice between decisions, is guided by the principle of maximization of the expected utility of the result.

Theory of the expected utility was offered at the end of 40th of the XX century and is the development of the neoclassical theory of the individual choice for cases with risk. For today, that approach of the expected utility can be considered conventional both in modern finances and in the economic theory as the whole.

According to this theory, each value of wealth ω corresponds to the certain level of benefits (utility) u(ω) that is provided to the person of the given value of wealth. According to the hypothesis of the expected utility, the person, who has a certain ideas of the probable distribution of value ω (we shall designate these distributions as P) will choose the decision that will bring the maximal value of the expected utility of that result

$$E_p[u(\omega)].$$

$E_p[u(\omega)]$ in the given formula designates the expected value.

As against the case of the determinacy under the risk, the choice of this or that decision would mean the choice of probable distribution of some aleatory variable (for example, the income).

According to John Von Neiman and Oscar Morgenstern, various probable distributions presuppose some preferences, in other words, the person may compare the utility of various distributions. And these preferences are rational: they have the following properties (meet the following axioms):

1) Completeness. For any two distributions (we shall designate them

as P_1 and P_2) the person can always point out the most preferable or consider them equivalent. Symbolically, this can be written as follows:

$$\forall P_1, P_2 : P_1 \succ P_2 \text{ or } P_1 \prec P_2, \text{ or } P_1 \approx P_2,$$

where the sign "$\succ$" means "better", "preferable", "$\prec$" – "worse" and "$\approx$" – "equivalent" accordingly.

2) Transitivity. If one distribution is better than the other, and the latter is better than the third one, then it can be said that the first distribution is better than the third one. In other words,

$$P_1 \succ P_2, P_2 \succ P_3, \Rightarrow P_1 \succ P_3.$$

3) Continuity. For any two distributions, one of which is better than the other, there is always a distribution that occupies the "intermediate" position, in other words, it is less preferable in comparison with the first one, but more preferable than the second one. This intermediate distribution can be presented in the form of linear combination of that first two:

$$P_1 \succ P_2 \Rightarrow \exists P_3 = \lambda P_1 + (1 - \lambda) P_2 : P_1 \succ P_3 \succ P_2, 0 < \lambda < 1.$$

If the person's preferences satisfy the given axioms, it is possible to strictly show that on the set of distributions there is such a function of preferences V (P) that for two different distributions, one of which is better than the other, the value of the function of preferences for the first one will be greater, than for the second one:

$$P_1 \succ P_2 \Rightarrow V(P_1) > V(P_2).$$

The formulated axioms do not allow us to make a conclusion on the real form of the preference function. The theory of the expected utility suggests postulating of one additional property of relations of the preference, the so-called linearity (independence).

4) For any two distributions, the first one will be more preferable than the second one only when any linear combination that includes the first distribution, will be more preferable than the similar linear combination, where the first distribution is replaced by the second one. Formally

$$\forall P_1, P_2, P_3, \lambda \in (0,1): P_1 \succ P_2 \Leftrightarrow \lambda P_1 + (1 - \lambda) P_3 \succ \lambda P_2 + (1 - \lambda) P_3.$$

From the last axiom it is possible to deduce the linearity property depending on probability of the preference function, in other words, the function of preferences, according to the theory of the expected utility, is the linear combination of probabilities of various results, and the ratios of this linear expansion will be the utility function of each results. For example, we have possible situations, each of which possesses the utility of the person, who makes decisions, equal to $u_1, u_2, ..., u_n$. If $p_1, p_2, …, p_n$ is the probabilities of each of possible situations, the linear by its probability function of preferences will look like:

$$V(p_1, p_2, ..., p_n) = p_1u_1 + p_2u_2 +... + p_nu_n.$$

Let's presume that the index that describes the result of the decision is that only one, for example, the size of wealth in each situation: ω_1, ω_2, ..., ω_n, a $u(\omega)$ - the utility function of wealth. Then the function of preferences will look like:

$$V(p_1, p_2, ..., p_n) = p_1u(\omega_1) + p_2u(\omega_2) +... + p_nu(\omega_n). \qquad (2.1)$$

Generally, when possible results are described not by the discrete (as in the given example), but by the continuous distribution P, the linear by its probability function of preference will be as follows:

$$V(P) = \int_\Omega u(\omega)P(d\omega), \qquad (2.2)$$

where, Ω - a probabilistic space,

$\int P(d\omega)$ - an integral of Lebegue.

If there is the denseness of distribution $p(\omega)$, the expression (2.2) can written down as follows:

$$V(P) = \int_{-\infty}^{\infty} u(\omega)p(\omega)d\omega. \qquad (2.2a)$$

All three expressions of the function of preferences (2.1), (2.2) and (2.2a) represent the V function as the expectation of the utility function. Thus, according to the theory of Neumann – Morgenstern, the individual make his choice under risk basing on maximization of the expected utility of result: the person chooses such a distribution P* from the set of alternatives, that

$$V(P^*) = \max_P V(p) = \max_P E_p[u(\omega)].$$

Despite of the tremendous progressive significance of the theory of the expected utility, it contains, in our opinion, the following drawbacks:

1) If the only measure of the risk are indexes of the standard deviation and variance, this presupposes the quadratic form of the utility function

$$u(\omega) = \alpha\omega - \beta\omega^2, \quad \alpha, \beta>0.$$

2) The variance may vary as times goes by.

3) When computing the variance, it is necessary to summarize statistical data obtained at various time intervals, and they are not comparable generally.

4) Distribution of the aleatory variable may not have the form of normal distribution.

All the above testify to the necessity of searching of new approaches.

In our view, the person, by choosing the decision from the set of alternatives, is guided by the principle of "minimum of risk – maximum of utility", i.e. by the principle of the minimum of the functional

$$S = \int_{t_1}^{t_2} (R - U)dt \rightarrow \min \text{ to } t_1 < t < t_2, \qquad (2.3)$$

where, R - the risk function,

U - the utility function,

S - we shall call the functional of the optimal behaviour of an investor (FOBI), and the integral - as an action of the investor. **Within the point of time interval [t_1, t_2], the person distributes available resources so that within this period of time the functional (2.3) was of minimum value.**

2.4. Updated portfolio theory

Any economic problem is reduced to the task of the best distribution of resources. Any economic subject – an individual, firm or government - face the problem: how to distribute the available resources - material, financial, manpower etc. The economic theory proceeds from the assumption that any subject, by making decision, strives to the greatest benefit, in other words, acts rationally. The same principle lays in the basis of decision-making on the best investments.

What is better, say, for the given investor: to keep his savings under the pillow, to place on the bank deposit, to lend money to the mother-in-law, to purchase shares of the firm X, or maybe better to sell them? The answer to this question is not obvious. Each of possible decisions has the set of measurements - advantages from different ways of investments, time of the investment and, at last, the risk connected with this or that decision.

The start of the modern portfolio theory was given by Harry

Markovitz' s revolutionary work in 1952. The results of Markovitz have been developed and supplemented by the works of James Tobin, William Sharp and other researchers.

Importance of these works for the development of modern economics and finance is underlined by the Nobel Prize in economics that was awarded to Harry Markovitz, James Tobin and William Sharp first of all, for the development of modern portfolio theories.

An investor should not necessarily choose some one decision; he may choose any combination of possible investments, distributing his wealth to various directions of investment.

Without loss of generality, it is possible to describe the dynamics of portfolio profitability consisting of i-type of assets by the form of the radius-vector $r = r(q_1, q_2, ..., q_i)$ with its constituents, which coincide with values of profitability of separate assets that make the portfolio. Hereinafter, we shall call them as components. The derivative r in time t

$$v = \frac{dr(q_1, q_2, ..., q_i)}{dt}$$

яis the profitability rate of change (PRC). Below, as it is accepted, we shall designate derivation in time by the dot over the letter $v = \dot{r}$.

Components q_1, q_2..., q_i will be called as profitability of the investment portfolio, or simply as profitability q_1, q_2..., q_i, and $\dot{q}_1, \dot{q}_2, ..., \dot{q}_i$ profitability rates of change.

Simultaneous assignment of all profitability and PRC completely determines the state of the investment portfolio and allows, in principle, predicting of its evolution in future. From the mathematical point of view, it means that by assigning all profitability q_1, q_2..., q_i and PRC $\dot{q}_1, \dot{q}_2, ..., \dot{q}_i$ at some moment time one can determine also values of the first derivatives in time depending on PRC $\ddot{q}_1, \ddot{q}_2, ..., \ddot{q}_i$

$$\ddot{r}(q_1, q_2, ..., q_i) = \frac{d}{dt} v(\dot{q}_1, \dot{q}_2, ..., \dot{q}_i) = \frac{d^2}{dt^2} r(q_1, q_2, ..., q_i).$$

Relations, which are connecting the first derivatives in time depending on the PRC with profitability, will be called *as equations of economic systems evolution.* In respect to q (t) functions, these are differential second-order equations, integration of which actually allows defining of these functions, i.e. "trajectory" of profitability investment portfolio changes.

After certain preparatory work, we shall formulate the principle of the investor's optimal behaviour: behaviour of each investor is characterized by the certain function:

$$L(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s, t)$$

or, in the brief, $L(q, \dot{q}, t)$. Evolution of the investor's tendencies satisfies the following condition: *presume that at certain moments of time $t=t_1$ and $t=t_2$ the system has determined coordinates $q^{(1)}$ and $q^{(2)}$*. Then, between these positions, L function changes in such a way that the functional

$$S = \int L(q, \dot{q}, t)dt \tag{2.4}$$

had the least probable value.

The fact that (2.4) contains only *q* and $\dot{q}$, the higher derivatives $\ddot{q}, \dddot{q}, \ldots$ confirm the assertion that the investor's behaviour can be fully described by assigning profitability and PRC.

Let's pass over to finding of the differential equations, which resolve the task of determining the minimum of the integral (2.4). For simplification of formulas, we shall presume that the portfolio consists of securities of only one kind. Thus, only one *q (t)* function should be determined.

Let's presume that q = q (t) is just that function, for which S has the minimum value. It means that S will increase if q (t) is replaced by any function of the type

$$q(t) + \delta q(t), \tag{2.5}$$

where $\delta q(t)$ - function that is small in all time intervals from t_1 to t_2 (it is called as the variation of *q(t)* function). As at $t = t_1$ and $t = t_2$, then all compared functions (2.5) should take on one and the same values of $q^{(1)}$ and $q^{(2)}$, and it should be:

$$\delta q(t_1) = \delta q(t_2) = 0. \tag{2.6}$$

Change of S during substitution of *q* to $(q + \delta q)$ is determined by the difference

$$\int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)dt.$$

Expansion of this difference to degrees δq and $\delta\dot{q}$ (in the integrand expression) is started from the terms of the first order. The required condition of the S minimality is the equality to zero of the set of these terms; it is called as the first variation of the integral. Thus, the principle of the investor's optimal behaviour can be written down as

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0 \qquad (2.7)$$

or, by making variation

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0.$$

Taking into account that $\delta \dot{q} = \frac{d}{dt} \delta q$, we shall integrate the second term in parts and receive:

$$\delta S = \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \cdot dt = 0. \qquad (2.8)$$

But due to condition (2.6), the first term in this expression disappears. The integral remains, but it should be equal to zero at arbitrary values of δq. This is possible in the event, when the integrand expression is identically equal to zero. Thus, we receive the equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

If there exists several degrees of freedoms of the optimal behaviour of the investor, *s* of various $q_i(t)$ functions should vary independent. It is obvious that we shall receive the *s* equations of the type

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad (i = 1, 2, ..., s). \qquad (2.9)$$

These are the required differential equations. From the mathematical point of view, equations (2.8) constitute the system of the *s* second-order equations for *s* of the unknown $q_i(t)$ functions. The general solution of such system contains *2s* of arbitrary constants.

2.5. Choice of the best portfolio and diversification

So, we shall consider decisions of the investor, who is in possession of wealth Q and is facing the problem - how to use this wealth.

Let's consider that choice between the current consumption and investments has already been made, and accordingly, Q is the cost that will not be consumed within the current period. We presume for simplicity that the investor considers the fixed planned time – the means Q are to be distributed for a certain period.

Our investor has *n* possibilities to use his means, each of which will bring, accordingly, $\xi_1, \xi_2, \ldots, \xi_n$ dollars of income per 1 $ of investments. The most important problems for decision-making is that values ξ_i, in general, are random ones, i.e. it is not known beforehand, what will be the income.

The principal supposition that is accepted by the authors of this work while analyzing this task, is that for the investor, when estimating alternative decisions, only two parameters are important:

First - expected profitability of investments

$$\bar{r}_1 = E\xi_i,$$

where E - expectation;

Second - square of the profitability rate of change $\left(\frac{dr}{dt}\right)^2$ as the index that describes the risk of the decision to be accepted.

2.5.1. Average profitability of the portfolio

Let's determine the average profitability of the portfolio (ξ_p) as the increase of wealth per unit of investments guarantied by the given portfolio by the moment of time that is considered as the planned time

$$\xi_p = \frac{Q_1 - Q_0}{Q_0},$$

where Q_0 - today's size of wealth,

Q_1 - size of wealth by the end of the period.

Profitability of the portfolio can be calculated as the weighted average profitability of investments of each assets included into the portfolio

$$\xi_p = \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n = \sum_{i=1}^{n} \xi_i x_i .$$

This expression can be presented in the vectorial form

$$\xi_p = \xi^T x,$$

where $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$

The expected profitability of the portfolio $\bar{r}_p$ is determined by formula of expectation of aleatory variables:

$$\bar{r}_p = E\xi_p = E(\xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n) =$$
$$= E(\xi_1 x_1) + E(\xi_2 x_2) + \ldots + E(\xi_n x_n).$$

By taking the determined values x_i (i = 1..., n) outside the sign of expectation, we shall receive

$$\bar{r}_p = E\xi_p = x_1 E\xi_1 + x_2 E\xi_2 + \ldots + x_n E\xi_n =$$
$$= x_1 \bar{r}_1 + x_2 \bar{r}_2 + \ldots + x_n \bar{r} = \sum_{i=1}^{n} x_i \bar{r}_i, \quad (2.10)$$

or in the vectorial form:

$$\bar{r}_p = x^T \bar{r}, \quad \text{где} \quad \bar{r} = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_n \end{pmatrix}.$$

Thus, that expected profitability of the investment portfolio is the weighted-average expected profitability of each asset of investments (in parts) included into the portfolio.

Degree of risk of the portfolio.

Profitability rate of change (PRC) of the portfolio is equal to:

$$\dot{\xi}_p = \dot{\xi}_1 x_1 + \dot{\xi}_2 x_2 + \ldots + \dot{\xi}_n x_n = \sum_{i=1}^{n} \xi_i x_i .$$

Let's square both parts of the equality and we shall receive value of risk of the portfolio:

$$\dot{\xi}_p^2 = \left(\dot{\xi}_1 x_1 + \dot{\xi}_2 x_2 + \ldots + \dot{\xi}_n x_n\right)\left(\dot{\xi}_1 x_1 + \dot{\xi}_2 x_2 + \ldots + \dot{\xi}_n x_n\right) = \sum_{i=1}^{n} \dot{\xi}_i x_i \cdot \sum_{i=1}^{n} \dot{\xi}_i x_i =$$

$$= \sum_{i=1}^{n} x_i^2 \dot{\xi}_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} x_i \cdot x_j \cdot \dot{\xi}_i \cdot \dot{\xi}_j .$$

2.5.2. Estimation of the average profitability, standard deviation and PRC on the basis of historical data

Use of historical data may be one of method of financial engineering estimation for average profitability and PRC. Some drawbacks of this approach should be underlined at once. First, historical method evaluates parameters of financial assets to the past (saying that the result in this case are *ex post* values - after observation), whereas the decision-making investor is interested in the future values (*ex ante*). Second, for computation, it is necessary have a series of observations over the actual value of profitability and profitability rate of change for several periods, and this information is frequently either not available (if we speak about re-emitted securities) or hard to be obtained, or contains significant distortions. All this is typically for the still insufficiently developed and closed market of the CIS countries. Third, economic values, which refer to different moments of time, are non-comparable at all.

Nevertheless, if it is necessity to estimate that average profitability and risk of some securities and application of more exact method is impossible, historical approach is the most effective.

Let's express as r_t profitability of some securities observed at the period *t*. We total of T observations (t = 1..., T). Then, the statistical estimation of the index of average profitability can be calculated as

$$\bar{r} = \frac{\sum_{t=1}^{T} r_t}{T}. \qquad (2.11)$$

Taking into account, that

$$\dot{r}_i(t) \cdot \dot{r}_j(t') = \delta_{tt'} \cdot \dot{r}_i(t) \cdot \dot{r}_j(t),$$

where $\delta_{tt'}$ – δ – function, $\delta_{tt'} = \begin{cases} 1 & t = t' \\ 0 & t \neq t' \end{cases}$,

we shall receive the square of PRC:

$$\left(\frac{dr}{dt}\right)^2 = \frac{\sum_{t=1}^{T} \left(\frac{dr_t}{dt}\right)^2}{T-1}. \qquad (2.12)$$

Let's presume that there is an information on profitability of two kinds of securities - shares of the joint-stock company "Yupiter" and the joint-stock company "Vostok" (table 6 and figure 10). It is necessary to form the portfolio consisting of two assets so that the its risk was minimum:

1) by using the method of variances;
2) by using the method based on the scalar product of rates of asset profitability change.

According to the traditional approach, the riskiness of the portfolio is calculated by formula:

$$\sigma_p := \sqrt{x_1^2(\sigma_1)^2 + (1-x_1)^2(\sigma_2)^2 + 2x_1(1-x_1)\rho_{12}\sigma_1\sigma_2},$$

where - standard deviation of the portfolio,
x_1 - densities of the first asset,
$(1 - x_1)$ - densities of the second asset,

- standard deviation of the first asset,

- standard deviation of the second asset,

Table 6. Historical data on profitability
of the JSC “Yupiter” and JSC “Vostok

Number of the period of time	Profitability of the JSC “Yupiter” asset	Profitability rate of change of the JSC “Yupiter” asset*	Profitability of the JSC “Vostok” asset	Profitability rate of change of the JSC “Vostok” asset	Scalar product of the PRC of JSC “Yupiter” and JSC “Vostok”*	Square of PRC of JSC “Yupiter”	Square of PRC of JSC “Vostok”
1	0.1108	-0.05277	0.00365	0.01885	-0.000995	0.002785	0.000355
2	0.05803	-0.03136	0.0225	0.03714	-0.001165	0.000983	0.001379
3	0.02667	0.03813	0.05964	0.05434	0.002072	0.001454	0,002953
4	0.0648	0.05934	0.11398	-0.0891	-0.005287	0.003521	0,007939
5	0.12414	-0.04682	0.02488	0.07497	-0.00351	0.002192	0,005621
6	0.07732	-0.03752	0.09985	-0.00658	0.000247	0.001408	4,33E-05
7	0.0398	0.0512	0.09327	-0.06743	-0.003452	0.002621	0,004547
8	0.091	-0.01742	0.02584	0.08893	-0.001549	0.000303	0,007909
9	0.07358	0.01345	0.11477	-0.0714	0.00096	0.000181	0,005098
10	0.08703	-0.00721	0.04337	0.02502	-0.00018	5.2E-05	0,000626
11	0.07982	-0.06336	0.06839	-0.02666	0.001689	0.004014	0,000711
12	0.01646	0.04704	0.04173	0.0083	0.00039	0.002213	6,89E-05
13	0.0635	-0.03488	0.05003	0.02524	-0.00088	0.001217	0,000637
14	0.02862	0.08632	0.07527	-0.01676	-0.001447	0.007451	0,000281
15	0.11494	-0.05135	0.05851	0.00948	-0.000487	0.002637	8,99E-05
16	0.06359	-0.00628	0.06799	0.03509	-0.00022	3.94E-05	0,001231
17	0.05731	-0.01344	0.10308	-0.07595	0.001021	0.000181	0,005768
18	0.04387	0.02907	0.02713	-0.02239	-0.000651	0.000845	0,000501
19	0.07294	-0.02828	0.00474	0.09787	-0.002768	0.0008	0,009579
20	0.04466		0.10261				

* Asset profitability rate of change is defined as the difference among the values of this asset for the (n -1) and n-period of time.

- correlation of the first and the second asset profitability.

The least risky will be the portfolio that includes 65 % of assets № 1 and 35 % of assets № 2 (fig. 11, a; fig. 12, a).

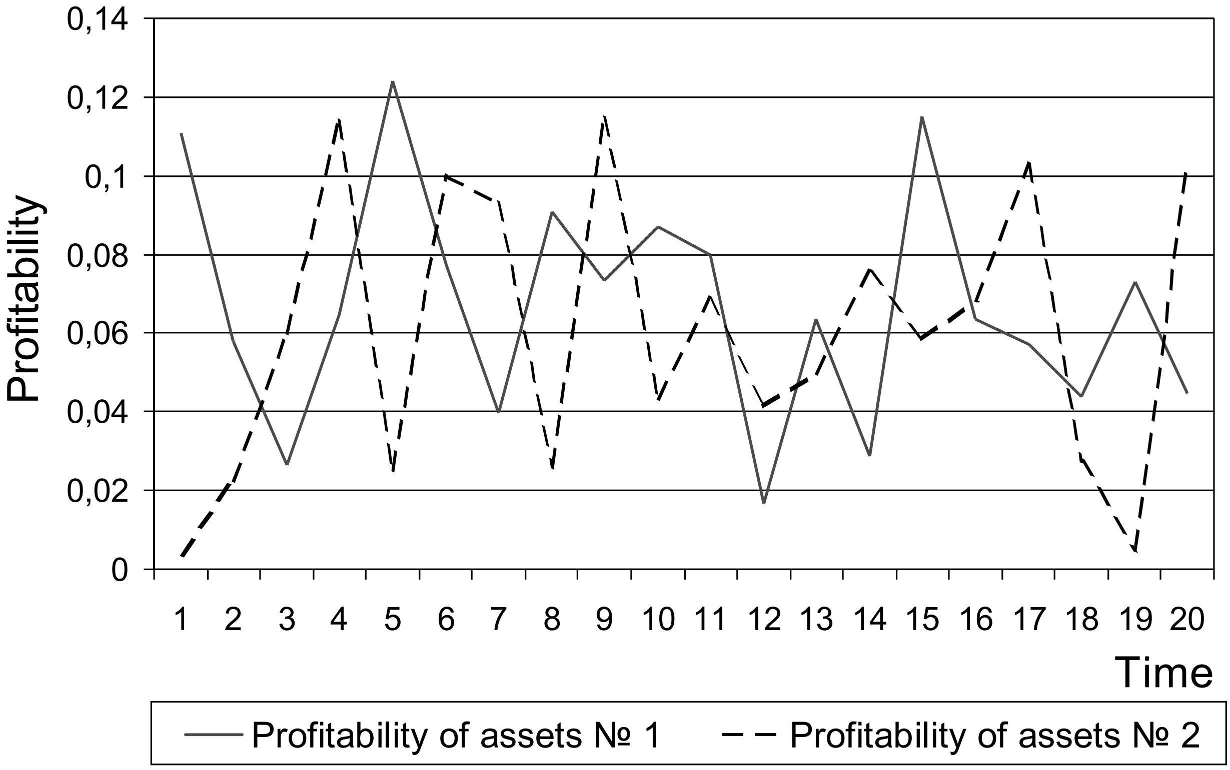


Fig. 10. Initial data

According to our offered method, the riskiness of the portfolio is calculated by formula:

,

where - riskiness of the portfolio,

v_1 - profitability rate of change of the first asset,

v_2 - profitability rate of change of the second asset,

x_1 - densities of the first asset,

$1 - x_1$ - densities of the second asset,

$\cos\angle(v_1, v_2)$ - cosine of the angle between the averaged rates of the change of profitability of the first and second assets.

Calculation made according to our method testifies that the least risky portfolio from two assets consists of 64 % of assets № 1 and 36 % of assets № 2 (fig. 11, b; fig. 12, b).

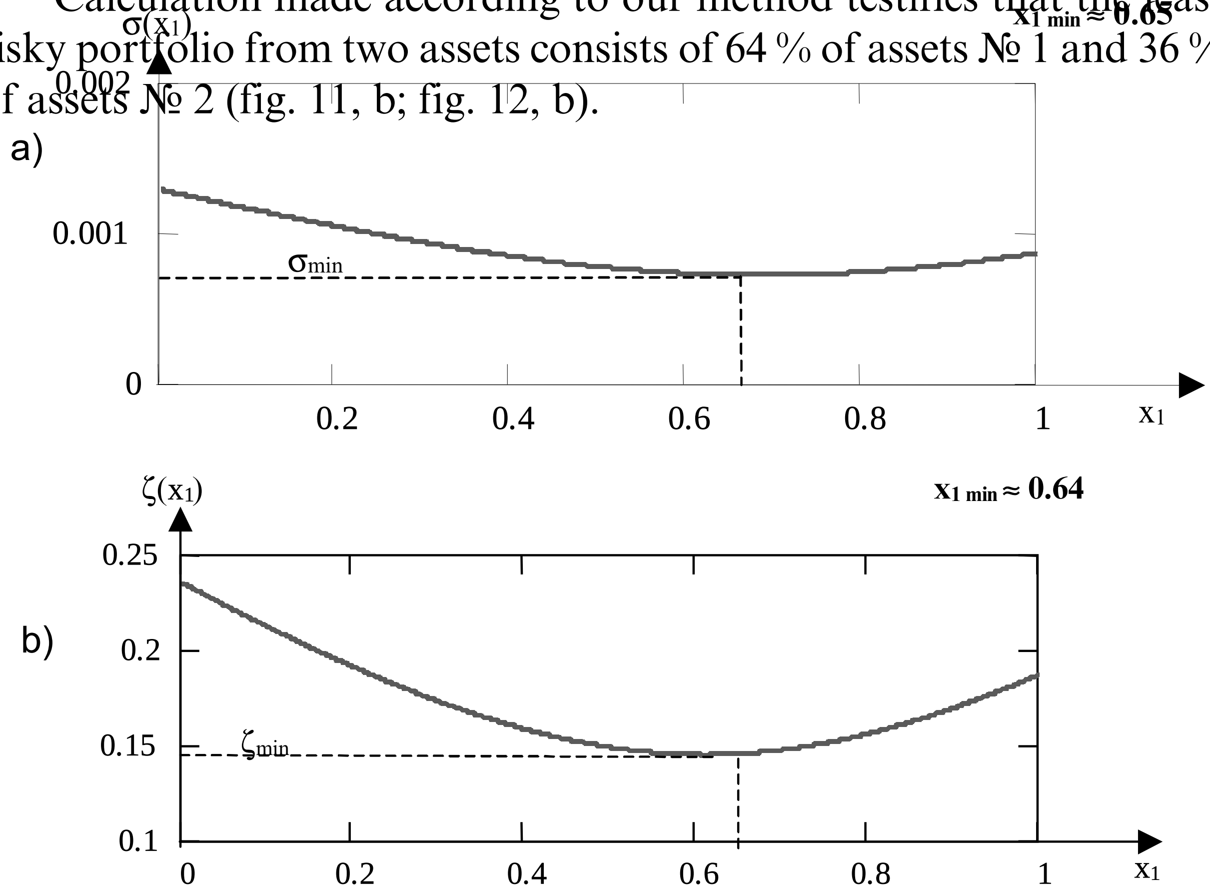


Fig. 11. Dependence of risk on the structure of the portfolio:
a) according to the existing theory;
б) according to the method offered by the authors

Using these data and formulas (2.11) - (2.12), we shall know that the estimation of that average profitability of shares of the joint-stock company "Yupiter" is equal 6,69 %, square of PRC - 0,035, for the joint-stock company "Vostok" - 6,00 % and 0,055 accordingly.

For computation of the scalar product of PRC of two assets, we use the formula

$$(\dot{r}_1, \dot{r}_2) = \frac{\sum_{t=1}^{T} \dot{r}_1 \cdot \dot{r}_2}{T-1}, \qquad (2.13)$$

where $\dot{r}_1, \dot{r}_2$ - RPC of values r_1, r_2.

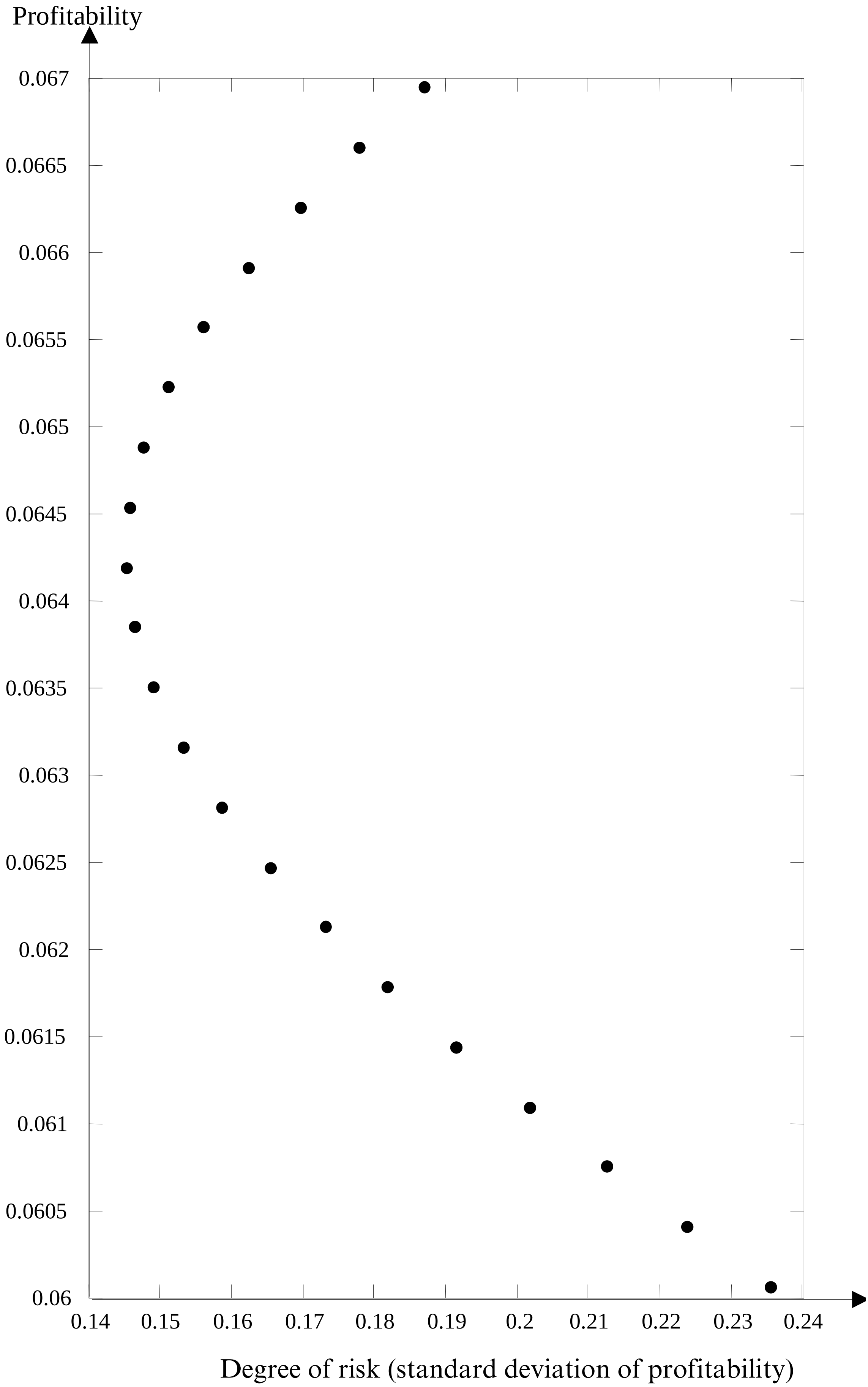


Fig. 12, a. Dependence of risk depending on profitability accornding to the existing theory

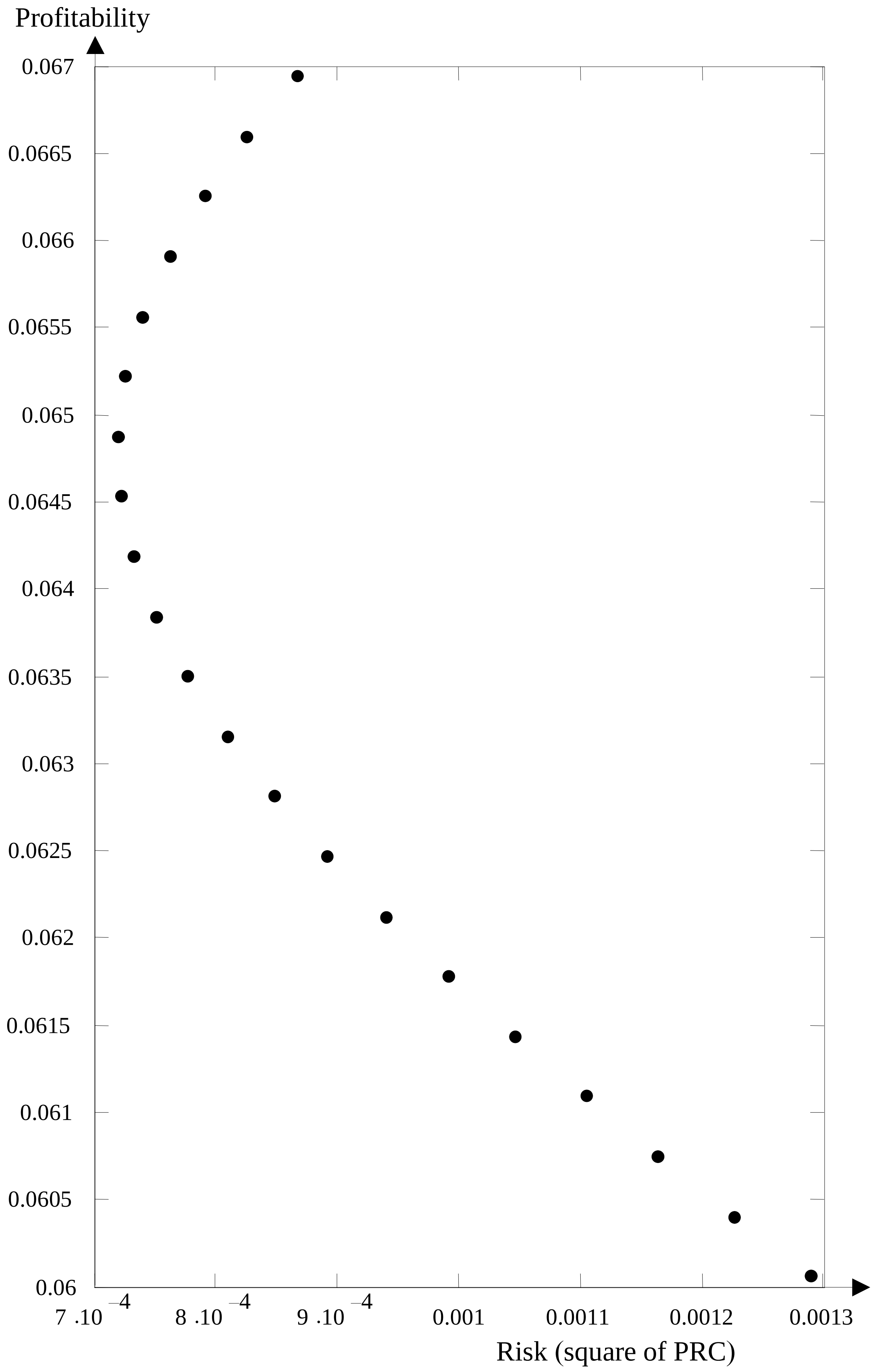


Fig. 12, b. Dependence of risk depending on profitability accornding to the theory offered by the authors

In our example, the scalar product PRC is equal to:

$$(\dot{r}_1, \dot{r}_2) = -0{,}0181.$$

Taking into account that $(\dot{r}_1)^2 = \|\dot{r}_1\|^2$, $(\dot{r}_2)^2 = \|\dot{r}_2\|^2$, it is possible to determine some angle $\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2)$ between the averaged vectors of rates of the change of profitability of the first and second assets, depending on the identity,

$$\cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2) \equiv \frac{(\dot{\bar{r}}_1, \dot{\bar{r}}_2)}{\|\dot{\bar{r}}_1\| \cdot \|\dot{\bar{r}}_2\|},$$

which we shall call as the averaged angle between the averaged vectors of rates of the change of profitability. In our case

$$\cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2) = \frac{-0{,}0181}{0{,}187 \cdot 0{,}2352} = -0{,}4115.$$

Once again, it is necessary to underline that indexes computed by formulas (2.11) - (2.13) represent only the rough estimates of historical values of the average profitability and square PRC. What is more, it is impossible to think that these estimations can somewhat precisely predict the future values of the considered indexes. At the same time, in practice, when there is no possibility of more exact estimation of indexes of profitability and risk of securities, this method may be effective.

As it is obvious from the received results, the PRC method of determination of investment portfolio with minimum risk gives approximately identical results with the method that is uses standard deviation as the criterion of the degree of risk. Each of them has their own boundaries of application. The authors hope that in the near future, their method will find its practical application.

2.5.3. Portfolio consisting of two assets

Let's presume that possibilities of investments are limited by two kinds of securities (already used as an example) that have been calculated by us on the basis of historical data of estimations and reflect the indexes of profitability and risk:

1) the first asset - shares of the joint-stock company "Yupiter", which are characterized by profitability ξ_1, the average (expected) value

of profitability $\bar{r}_1$ is equal to:

$$\bar{r}_1 = E\xi_1 = 6{,}69\%,$$

the value of risk $\dot{r}_1^2$ is equal to:

$$\dot{r}_1^2 = 0{,}034;$$

2) The second asset (shares of the joint-stock company "Vostok") guarantees the random profitability ξ_2, the expected value of which is equal to:

$$\bar{r}_2 = E\xi_2 = 6{,}00\%,$$

PRC is equal to

$$\dot{r}_2^2 = 0{,}0553.$$

Let's presume that x_1 - the share of the investor's wealth that is invested into the shares of the joint-stock company "Yupiter", and x_2 - the share of the investor's wealth that is invested into the "Vostok". We may calculate the average profitability and the risk function of the portfolio:

$$\bar{r}_p = x_1\bar{r}_1 + x_1\bar{r}_1 = x_1 \cdot 6{,}69 + x_2 \cdot 6{,}00, \qquad (2.14)$$

$$\dot{r}_p^2 = x_1^2(\dot{r}_1)^2 + x_2^2(\dot{r}_2)^2 + 2x_1x_2(\dot{r}_1\,\dot{r}_2). \qquad (2.15)$$

Let's remind, that

$$\dot{r}_1\,\dot{r}_2 = (\dot{\xi}_1, \dot{\xi}_2) = -0{,}018133,$$

whence,

$$\dot{r}_p^2 = x_1^2 \cdot (0{,}034898)^2 + x_2^2 \cdot (0{,}055336)^2 + 2x_1x_2 \cdot (-0{,}018133).$$

For example, if the investor has made a decision to distribute the investments equally between in the first and second assets of the portfolio, i.e. x_1 = 0.5 = 50 % and x_2 = 0.5 = 50 %, then

$$\bar{r}_p = 0.5 \text{ x } 6{,}69 + 0.5 \text{ x } 6{,}00 = 6{,}34\%,$$

$$\dot{r}_p^2 = (0.5)^2 \cdot 0{,}0348 + (0.5)^2 \cdot 0{,}0553 - 0{,}0183 \cdot 0.5 \cdot 0.5 = 0{,}0179.$$

2.5.4. Effect of diversification

Let's remind, that if $(\dot{r}_1, \dot{r}_2)$ is the scalar product of PRC of some two securities, then

$$(\dot{r}_1, \dot{r}_2) = \|\dot{r}_1\| \cdot \|\dot{r}_2\| \cdot \cos\angle(\bar{\dot{r}}_1, \bar{\dot{r}}_2),$$

where $\|\dot{r}_1\|, \|\dot{r}_2\|$ - modules of rates of change of profitability of the first and second assets,

$\angle(\bar{\dot{r}}_1, \bar{\dot{r}}_2)$ - averaged angle between the vectors of rates of change of profitability.

Therefore, the formula for computation of the square of the portfolio profitability rate of change consisting of 2 assets can be written as

$$\dot{r}_p = x_1^2 \dot{r}_1^2 + x_2^2 \dot{r}_2^2 + 2 x_1 x_2 \|\dot{r}_1\| \cdot \|\dot{r}_2\| \cdot \cos\angle(\bar{\dot{r}}_1, \bar{\dot{r}}_2).$$

Thus, the common the risk of the portfolio depends on the value of risk of the assets included into the portfolio $\dot{r}_i^2$, on the share of each asset in the portfolio x_i and the ratio that describes that relation between that values of profitability of assets included into the portfolio $\cos\angle(\bar{\dot{r}}_1, \bar{\dot{r}}_2)$.

The effect of diversification - distribution of investments between various directions – is at during search of optimal sizes of investments into the various assets the investor may control the riskiness of the portfolio, by choosing such a value of $\dot{r}_p^2$ from all possible that answers his preferences. Possibilities of the decrease of the portfolio risk depends on closeness of correlation between the profitability of various investment decisions (in our example – values of $\cos\angle(\bar{\dot{r}}_1, \bar{\dot{r}}_2)$). We should remind that the cosine of the angle might vary within the limits from -1 to 1. The value of $\cos\angle(\bar{\dot{r}}_1, \bar{\dot{r}}_2) = -1$ means a perfect negative correlation: if the profitability rate of change of one asset is increased, the profitability rate of change of the second is proportionally reduced. In case of $\cos\angle(\bar{\dot{r}}_1, \bar{\dot{r}}_2) = +1$ both assets are characterized by the perfect positive correlation: any increase in the profitability rate of change of one of them will proportionally reduces the second profitability rate of change. If $\cos\angle(\bar{\dot{r}}_1, \bar{\dot{r}}_2) = 0$, then the profitability rate of change of one asset is in no way connected with the profitability rate of change of the second one.

2.5.5. Portfolio with the minimum risk

Let's presume that the investor's purpose is the choice of the portfolio with minimum probable risk, in other words, it is necessary to choose such x_1 and x_2 so that that value of risk of the portfolio $\dot{r}_p^2$ was the least. This task can be easily solved analytically. First of all, we shall remark that for the portfolio consisting of 2 assets there are always some budget limitation to be fulfilled:

$x_1 + x_2 = 1$, - the sum of share is equal to one.

The value of x_2 can be express through x_1

$$x_2 = 1 - x_1.$$

Let's designate

$$x = x_1,$$
$$1 - x = x_2.$$

The task of the choice of the portfolio with the least risk can be written down as:

$$\min_x \left\{x^2 \dot{r}_1^2 + (1-x)^2 \dot{r}_2^2 + 2x(1-x)\dot{r}_1\dot{r}_2 \cdot \cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2)\right\} \tag{2.16}$$

Let's write the condition of the first order for this task (we shall take the x - derivative and equate it with zero)

$$-\frac{2x\dot{r}_1^2 - 2(1-x)\dot{r}_2^2 + 2x(1-2x)\dot{r}_1 \cdot \dot{r}_2 \cdot \cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2)}{2\sqrt{x^2\dot{r}_1^2 + (1-x)^2\dot{r}_2^2 + 2x(1-x)\dot{r}_1 \cdot \dot{r}_2 \cdot \cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2)}} = 0,$$

whence, under condition that the denominator is not equal to zero, we shall receive:

$$x^* = \frac{\dot{r}_2^2 - \dot{r}_1 \cdot \dot{r}_2 \cdot \cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2)}{\dot{r}_1^2 + \dot{r}_2^2 - 2\dot{r}_1 \cdot \dot{r}_2 \cdot \cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2)}. \tag{2.17}$$

The formula (2.17) enables to determine the portfolio with the minimum risk (square PRC). Let's consider some special cases.

1. Let's presume that $\cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2) = -1$, then

$$x^* = \frac{\dot r_2^2 + \dot r_1 \dot r_2}{\dot r_1^2 + \dot r_2^2 + 2\dot r_1 \dot r_2} = \frac{\dot r_2(\dot r_2 + \dot r_1)}{(\dot r_1 + \dot r_2)^2} = \frac{\dot r_2}{\dot r_1 + \dot r_2},$$

$$1 - x^* = \frac{\dot r_1}{\dot r_1 + \dot r_2}.$$

In this case, the risk of the portfolio is equal to zero

$$\dot r_p^2 = \left(\frac{\dot r_2}{\dot r_1 + \dot r_2}\right)^2 \dot r_1^2 + \left(\frac{\dot r_1}{\dot r_1 + \dot r_2}\right)^2 \dot r_2^2 - 2\frac{\dot r_1}{\dot r_1 + \dot r_2} \cdot \frac{\dot r_2}{\dot r_1 + \dot r_2} \dot r_1 \dot r_2 = 0.$$

In other words, if profitability of the first asset will decrease, for the portfolio as the whole, this will be completely compensated by the growth of profitability of the second asset.

2. In the case, when $\cos\angle(\dot r_1, \dot r_2) = 0$, i.e. there is no correlation between the PRC of the first and second asset, the portfolio with the least risk is chosen as follows:

$$x^* = \frac{\dot r_2^2}{\dot r_1^2 + \dot r_2^2}, \quad 1 - x^* = \frac{\dot r_1^2}{\dot r_1^2 + \dot r_2^2}.$$

The square of PRC of such portfolio will be equal to $\dot r_p^2 = \dot r_1 \dot r_2$. If $\dot r_1 < 1$ and $\dot r_2 < 1$, the risk of the portfolio will be smaller, than the risk of each separate assets.

3. When $\cos\angle(\dot r_1, \dot r_2) = 1$, the optimal portfolio is chosen by the following way:

$$x^* = \frac{\dot r_2^2}{\dot r_2 - \dot r_1}, \quad 1 - x^* = \frac{-\dot r_1}{\dot r_2 - \dot r_1}.$$

The risk of such portfolio is also equal to zero:

$$\dot r_p^2 = \left(\frac{\dot r_2}{\dot r_2 - \dot r_1}\right)^2 \dot r_1^2 + \left(\frac{-\dot r_1}{\dot r_2 - \dot r_1}\right)^2 \dot r_2^2 + 2 \cdot \frac{\dot r_2}{\dot r_2 - \dot r_1} \cdot \frac{(-\dot r_1)}{\dot r_2 - \dot r_1} \cdot \dot r_1 \cdot \dot r_2 = 0,$$

but there are exist an important difference from the case, where $\cos\angle(\dot r_1, \dot r_2) =$ -l, when optimal investments into each of the assets were positive. In this case, either x^* or 1 - x^* is less than zero (and, if $x^* < 0$, (1 - x^*) > 1, and on the contrary). Negative investments mean short sale, when the borrowed assets borrowed is sold with obligation of subsequent return. Therefore, in case of $\cos\angle(\dot{r}_1, \dot{r}_2) > 0$ to receive the portfolio with minimum risk, it is necessary to sell one of the assets and invest all available and received means obtained due to short sale into the second asset.

2.5.6. Hedging

The above results allow to draw a very important conclusion: the higher the degree of correlation between PRC of two assets, the more possibilities to reduce the risk exist by way of changing combination of investments into these assets (shaping of the portfolio), in other words, the more effective the diversification undertaken with the purpose reducing the risk. This fact is the basis of the hedging strategy.

Hedging represents the strategy of reducing the risk, which is used by the investor for finding guarantee against possible losses connected with investments into some assets by simultaneous investments into other assets. The PRC of the latter is connected by negative dependence with the PRC of the first.

As an example, let's consider the same task of choosing portfolio, but changed a little.

Let's presume that the investor possesses one unit of some assets, which will bring him ϖ units of net profit within the planned time. We shall designate as x the size of investments into this asset: x = 1.

Profitability ϖ is an aleatory variable. We shall assume, that it may be both positive and negative. Presume that the investor wishes to secure himself against the risk of loss of the cost of his asset (in those cases, when ϖ will be negative. For this purpose, he invests means into another asset, the PRC of which $\dot{r}_\eta$ — is also random, but is connected by negative statistical dependence with the PRC of the first asset. We shall designate as h the size of investments into the second asset. Total expected profitability of investments (portfolio) will be equal to:

$$\bar{r}_p = E\varpi \times 1 + E\eta \times h = \bar{r}_\varpi + h\bar{r}_\eta .$$

The risk of the portfolio will be equal to:

$$\dot{r}_p^2 = \dot{r}_\varpi^2 + h^2 \dot{r}_\eta^2 + 2h(\dot{\varpi}, \dot{\eta}). \tag{2.18}$$

The risk will be minimum, if basing on conditions of the first order of the function minimum (2.18), the equation is true

$$2h\dot{r}_{\eta}^{2}+2(\varpi,\dot{\eta})=0$$

or

$$h=-\frac{(\dot{r}_{\varpi},\dot{r}_{\eta})}{\dot{r}_{\eta}^{2}}=-\frac{\dot{r}_{\varpi}\dot{r}_{\eta}\cos\angle(\dot{\bar{r}}_{\varpi},\dot{\bar{r}}_{\eta})}{\dot{r}_{\eta}^{2}}=-\cos\angle(\dot{\bar{r}}_{\varpi},\dot{\bar{r}}_{\eta})\cdot\frac{\dot{r}_{\varpi}}{\dot{r}_{\eta}}, \qquad (2.19)$$

where $\cos\angle(\dot{\bar{r}}_{\varpi},\dot{\bar{r}}_{\eta})$ - cosine of the averaged angle between vectors of rates of change of profitability - $\dot{r}_{\varpi}$ and $\dot{r}_{\eta}$.

Value h, calculated by formula (2.19), shall be called as coefficient of hedging with minimum risk. We should remark that, $\cos\angle(\dot{\bar{r}}_{\varpi},\dot{\bar{r}}_{\eta})=-1$,then h = 1, in other words, hedging will ensure the minimum risk, if each unit of means in the portfolio invested into the first asset will correspond to equally one unit of investments into the asset used for hedging.

2.5.7. Graphical illustration

Presuming that investors, when making decisions, are guided only by the average profitability and risk measured by the square of PRC, we can use figure 13 as illustration of portfolio investments, where the vertical axis is the average profitability, and horizontal - the risk, which is considered by us:

in case a) - standard deviation of profitability;

in case b) - square of PRC.

Points Ю and M correspond to assets, which we have chosen as an example (shares of the joint-stock company "Yupiter" and the joint-stock company "Vostok"). We shall determine, what will be the average profitability and square of PRC of several variants of the portfolio.

Profitability and square of PRC of portfolios is computed by formulas (2.14) and (2.15), accordingly. The results are given in figure 13 and in table 8.

In figure 13, on the vertical axis we lay off the average (expected) profitability of financial assets, on the horizontal axis – the square of profitability rate of change (risk). The point Ю (for the joint-stock company "Yupiter") corresponds to the securities that provide average returns at the level of 6.69 % annual at the value of square of profitability rate of change equal to 0.03489. Parameters of the

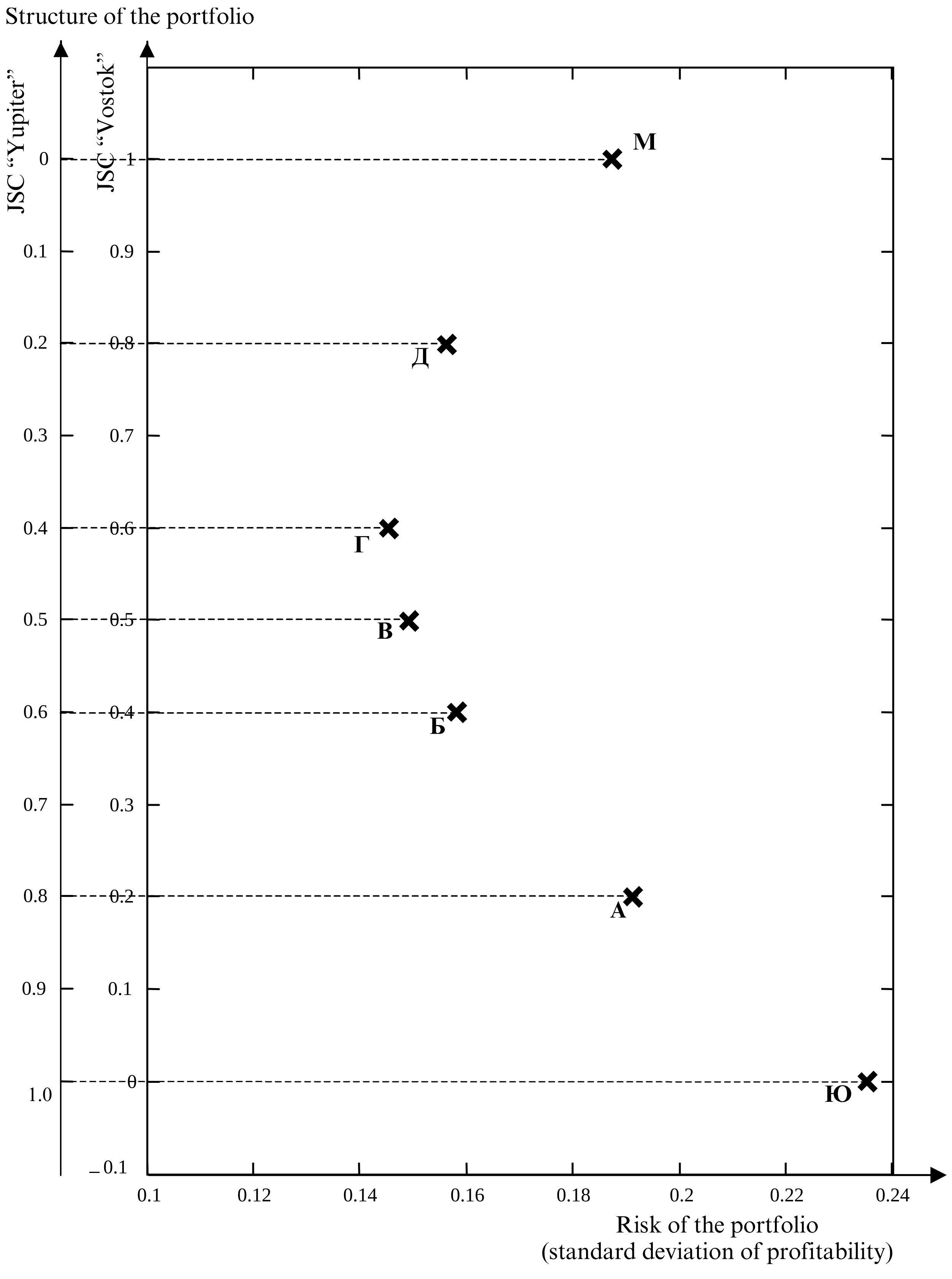


Fig. 13,a. Portfolio of two assets (traditional approach)

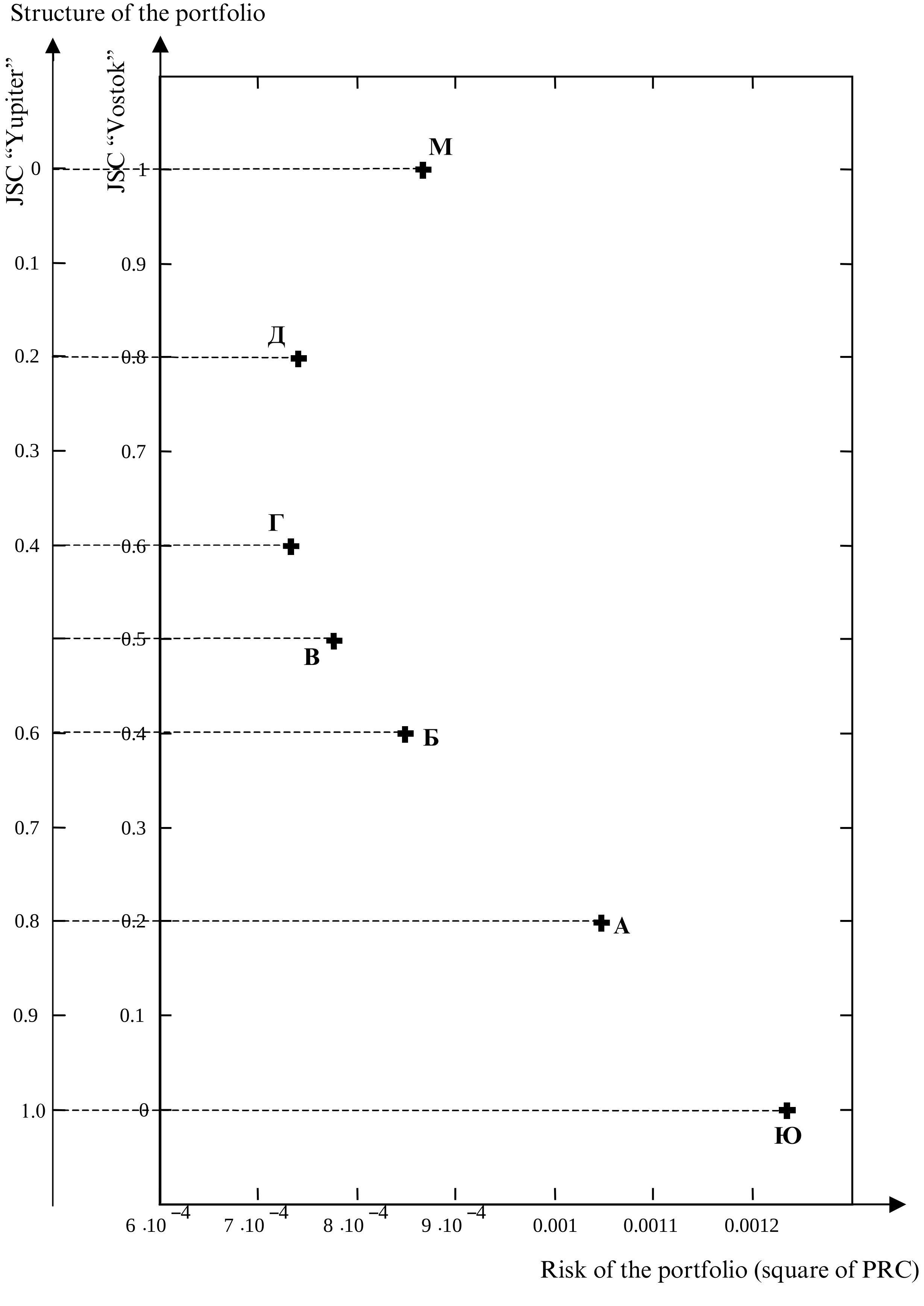


Fig. 13,b. Portfolio of two assets (approach offered by the authors)

Table 8. Possible portfolios consisting of shares of the joint-stock company “Yupiter” and the joint-stock company “Vostok”

	Portfolios						
	Ю	А	Б	В	Г	Д	М
JSC “Yupiter”	0	0,2	0,4	0,5	0,6	0,8	1
JSC “Vostok”	1	0,8	0,6	0,5	0,4	0,2	0
$I*X_{Ю}$	0	0,013389	0,026778	0,033472	0,040166	0,053555	0,0669
$(1-I)*X_{B}$	0,0600	0,04805	0,036037	0,030031	0,024025	0,012012	0
Profitability	0,0600	0,061438	0,062815	0,063503	0,064191	0,065568	0,0669
Risk according to the suggested theory*	0,235	0,191	0,158	0,149	0,145	0,156	0,187
Risk according to the existing theory*	$1{,}288\cdot10^{-3}$	$1{,}045\cdot10^{-3}$	$8{,}466\cdot10^{-4}$	$7{,}756\cdot10^{-4}$	$7{,}312\cdot10^{-4}$	$7{,}383\cdot10^{-4}$	$8{,}65\cdot10^{-4}$

$\frac{dx_{Ю}}{dt}$ (JSC “Yupiter”)	0,034898
$\frac{dx_{B}}{dt}$ (JSC “Vostok”)	0,055336
$X_{Ю}$ (JSC “Yupiter”)	0,066944
X_{B} (JSC “Vostok”)	0,060062
$\frac{dx_{Ю}}{dt} * \frac{dx_{B}}{dt}$	-0,01813
$var(X_{Ю})$	0,000865
$var(X_{B})$	0,001288
$cov(X_{Ю},X_{B})$	-0,0003

* Values of risk computed by two methods differ in absolute expression, i.e. in both case, different “units of measurement” of risk are used.

securities that are characterized by the point M (for the joint-stock company "Vostok") are equal to 6.00 and 0.055336 accordingly.

Points А, Б, В, Г, Д, Е, Ж correspond to combinations of risk and profitability ensured by various portfolios comprising assets of the joint-stock company "Yupiter" and the joint-stock company "Vostok".

It is possible to draw a conclusion that all possible portfolios lay on the curve M -А-Б-Г-Д-Ю. However, this relates only to the case, when $x_1 \geq 0$ and $x_2 \geq 0$. For comparison: figure 13, a shows the portfolio consisting of two assets (traditional approach).

2.5.8. Admissible combinations of risk and profitability at different degree of statistical correlation of assets included into the portfolio

Let's presume that there is a portfolio of two assets: in this portfolio, x_1 and x_2 designate the share from the total investments of each asset, $\bar{r}_1$ and $\bar{r}_2$ - expected profitability, $\dot{r}_1$ and $\dot{r}_2$ - PRC of the first and second asset, accordingly. The average profitability and risk of the portfolio is determined as:

$$\bar{r}_p = x_1\bar{r}_1 + x_2\bar{r}_2, \tag{2.20}$$

$$\dot{r}_p^2 = x_1^2 \cdot \dot{r}_1^2 + x_2^2 \cdot \dot{r}_2^2 + 2 \cdot x_1 \cdot x_2 \cdot \|\dot{r}_1\| \cdot \|\dot{r}_2\| \cdot \cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2). \tag{2.21}$$

Let's consider, using the figure, cases, when that rates of change of profitability of assets, included in the portfolio, are interconnected by absolute positive dependence $\cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2) = 1$, by absolute negative correlation $\cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2) = -1$, and also that situation, when there is no statistical correlation between the rates of change of profitability of two assets $\cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2) = 0$.

If $\cos\angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2) = 1$, the square of PRC of the portfolio is equal to:

$$\dot{r}_p^2 = x_1^2\dot{r}_1^2 + x_2^2\dot{r}_2^2 + 2x_1x_2\|\dot{r}_1\| \cdot \|\dot{r}_2\| = (x_1\dot{r}_1 + x_2\dot{r}_2)^2. \tag{2.22}$$

The equations (2.20) and (2.22) determine correlation between $\bar{r}_p$ and $\dot{r}_p$ at change of x_1 and x_2 parameters (we shall remind that $x_1 = 1 - x_2$). Thus, combinations of risk and income for various portfolios lay on the right line that passes through points *a* and *b* (fig. 14).

In figure 14, points a and b characterize combinations of risk and profitability of two assets included into the portfolio.

At value of $\cos\angle(\dot{r}_1, \dot{r}_2) = 1$ all possible portfolios composed of these are located along the right line *ab*.

At $\cos\angle(\dot{r}_1, \dot{r}_2) = -1$ all possible portfolios lay on the rays *cb* and *ca*. Dashed portions of the right line correspond to the short sale of one of the assets. At $0 < \cos\angle(\dot{r}_1, \dot{r}_2) < 1$ accessible combinations of risk and profitability are characterized by the parabola passing through the points *a* and *b*.

At $\cos\angle(\dot{r}_1, \dot{r}_2) = -1$,

$$\dot{r}_p^2 = x_1^2\dot{r}_1^2 + x_2^2\dot{r}_2^2 - 2x_1x_2\dot{r}_1\dot{r}_2 = (x_1\dot{r}_1 - x_2\dot{r}_2)^2. \qquad (2.23)$$

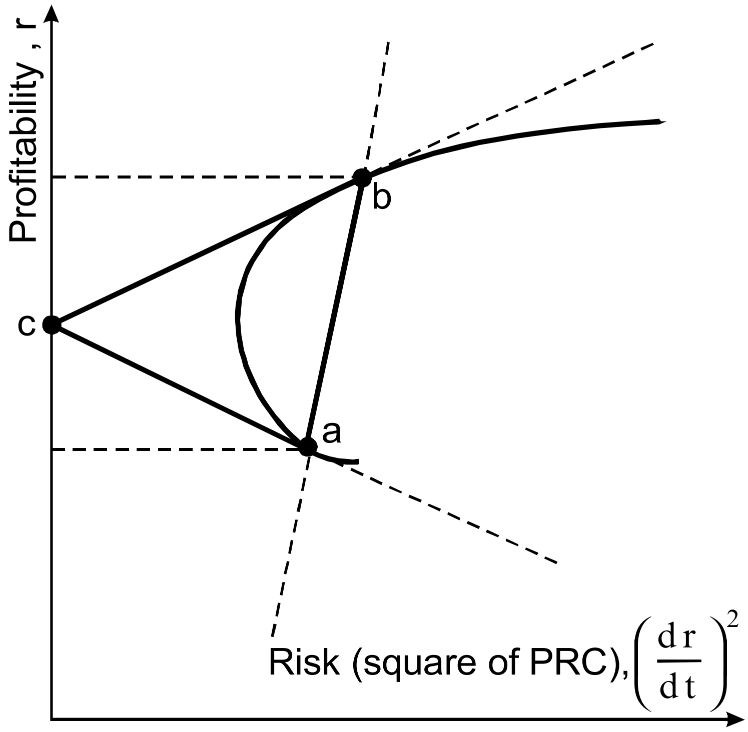


Fig. 14. Combinations of risk and income for two assets of the portfolio

Combinations of risk and income, appropriate to this case, are the points, which are laying on the rays *ca* and *cb* (fig. 14). Segments of *ca* and *cb* correspond to combinations of PRC and profitability under condition

$$x_1 \geq 0 \text{ и } x_2 \geq 0.$$

At last, if $\cos\angle(\dot{r}_1, \dot{r}_2) = 0$, all possible variants are reflected by the curve that is parametrically determined by equation (2.20) and equation

$$\dot{r}_p^2 = x_1^2 \dot{r}_1^2 + x_2^2 \dot{r}_2^2. \qquad (2.24)$$

Figure 14 allows to trace an important regularity: the less the value of (the closer it is to -1), the lower level of risk can be attained, and at $\cos \angle(\dot{\bar{r}}_1, \dot{\bar{r}}_2) = -1$ there exists the portfolio with the zero risk (point *c*). This fact has already been proved by us above and is used in the strategy of hedging.

2.6. Effectiveness of the investment portfolio

Let's return to the analysis of a more real situation, when the investor has the possibility to choose not from two, but from larger number of assets, each of which is characterized by the indexes of profitability and risk (points 1, 2, 3 and etc. in figure 15). Now the investor can choose any assets (any number) for inclusion into his portfolio. Curves 1-2, 2-4 and etc., reflect combinations of risk and income, which can be received by the investor by choosing the portfolio consisting exclusively from the first and the second asset, the second and the fourth, and so on. However, possibilities are not limited to this. Let's presume that the point A corresponds to the portfolio, 50 % of which are investments into the first asset and 50 % - into the second. Point B is the portfolio that represents the combination 50 : 50 from the first and the fifth assets. Then, the curve that passes through points A and B are values of risk and income appropriate to all possible combinations of the portfolio A and the portfolio B. For example, let's presume that the point O is the point obtained by combination in the common portfolio of 25 % of investments into the portfolio A and 75 % of investments into the portfolio B.

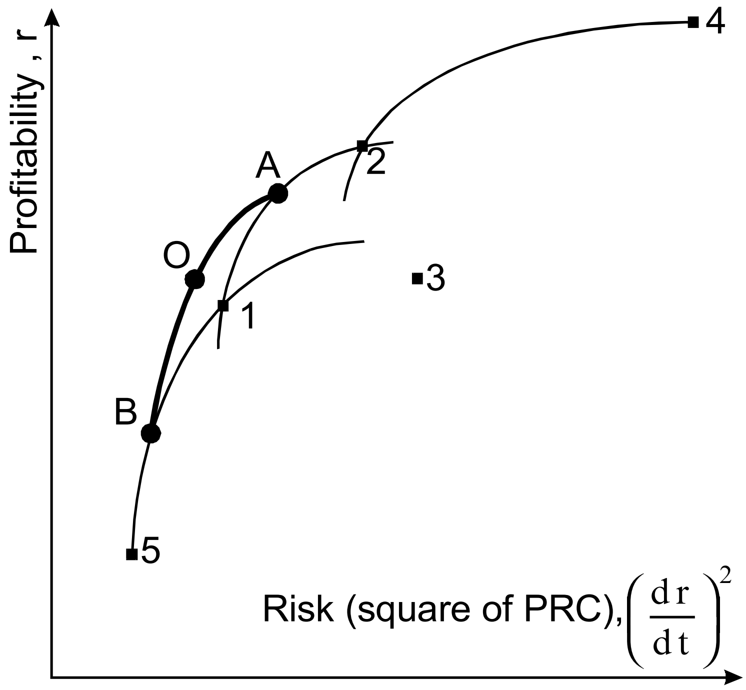


Fig.15. Possible combinations of investments

This would mean that the common portfolio contains:
1-st asset 0.5 x 0.25 = 0.125 = 12.5 %;
2-nd asset 0.5 x 0.25 + 0.5 x 0.75 = 0.5 = 50 %;
3-rd asset 0.5 x 0.75 = 0.375 = 37.5 %.

In figure 15, parabolas that are passing through points 1 and 2, 1 and 5, 2 and 4, characterize the risk and profitability of the portfolios formed from these assets, accordingly. Parabola AOB is the portfolios composed of various combinations of portfolios A and B. As possibilities of various portfolios formation are not limited, the effective set is convex.

All possible combinations of investments will form the area of admissible values of risk and income (dark area in figure 16). Obviously, (see fig. 15) that the area of admissible values is convex.

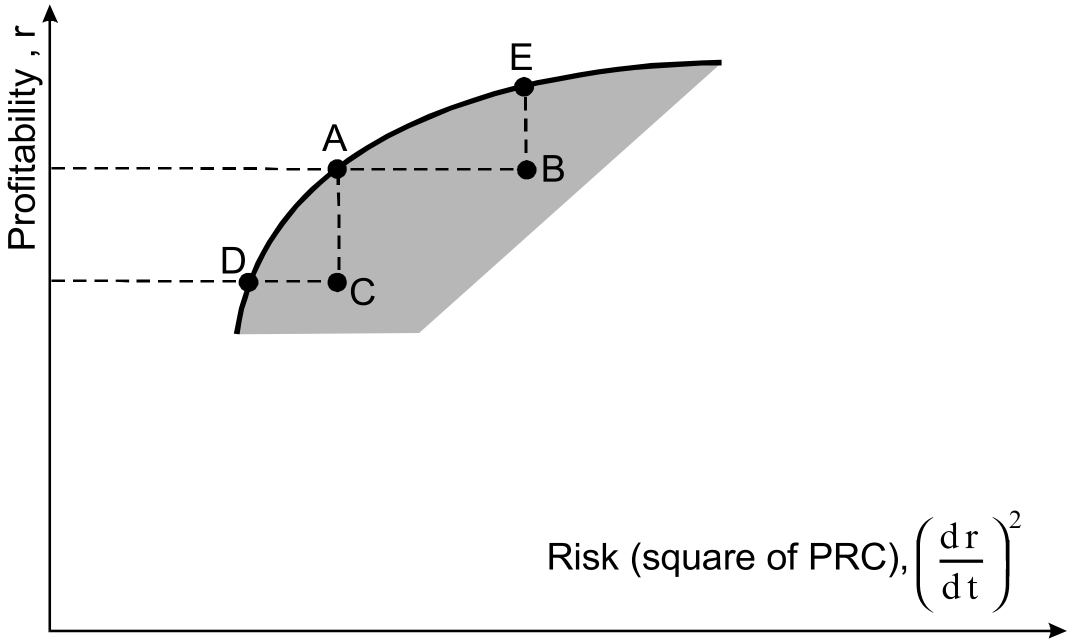


Fig. 16. Effective set and boundaries of effectiveness.
Points A, D, and E correspond to the effective portfolios.
C and B are not effective portfolios

Let's consider admissible portfolios from the point of view of their attractiveness for the investor, who is not inclined to risk. The person, wwho is not inclined to risk, will always prefer a smaller risk at the identical expected income and, accordingly, will always try to get the greater value of average returns at the same level of risk. Therefore, for the investor, who is not inclined to risk, the portfolio A, for example, is always better than the portfolio B or the portfolio C. Although portfolios A and B prove the same expected income, the portfolio B is more risky. Similarly, although portfolios A and C have the same degree of risk, portfolio C is less favourable from the point of view of its profitability.

An effective portfolio is a portfolio that provides the best level of the expected income at the given level of risk and the lowest risk at the given level of income. According to this definition, portfolios A, Д, E in figure 16 are effective, and portfolios B and C are not effective.

Apparently, if there exists the set of alternatives for investments (a set of assets), and the set of effective portfolios is also existing. In figure 16, the set of effective portfolios are the portfolios that are located on the boundary of admissible area. The set of effective portfolios is called the boundary of effectiveness or that effective set.

If possibilities of investments are limited only by some set of risky assets, i.e, attainable combinations of risk and income are limited by the area of admissible values of risk and expected profitability, the investor's choice is determined by his individual preferences (fig. 17).

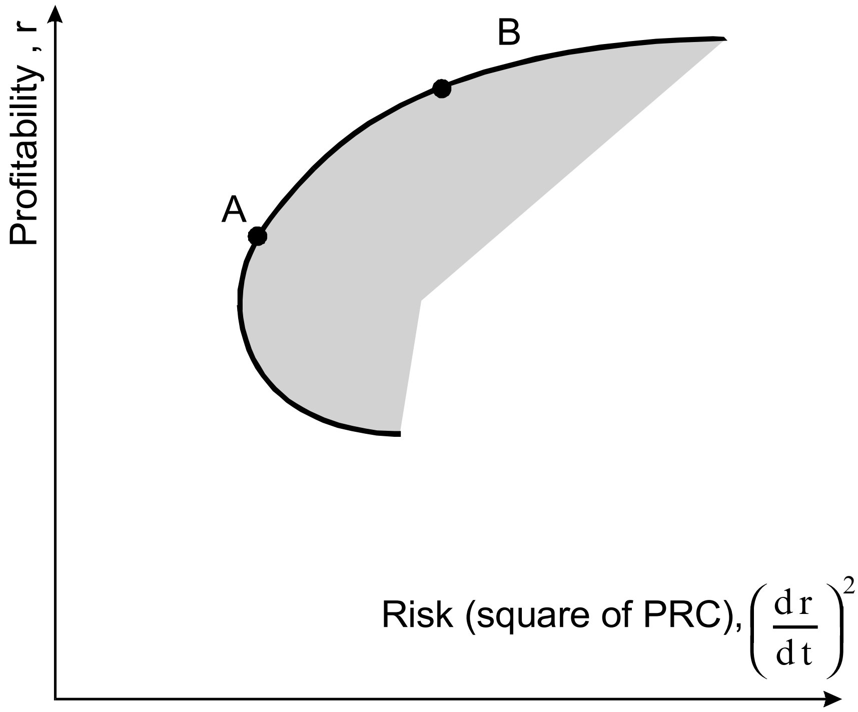


Fig. 17. Optimal choice from the admissible set of risky assets

The investor's choice among admissible values of risk and expected profitability is determined by his individual relation to preference. Relatively not inclined to the risk ("conservative") investor (point A) will strive for a smaller risk by sacrificing the income. The less non-inclined to risk ("aggressive") investor may resort to additional risk for the sake of the increase of income (point B). Common for all investors will be only the desire to form an effective portfolio, the expected profitability and square of PRC of which lays on the boundary of the admissible area.

2.7. Risk-free rate of profitability

Let's presume that in the market, in addition to risky investments, there is a possibility of financial investments, which guarantee the receipt of the certain income. In other words, let's presume that there is risk-free rate of profitability. In reality, it is difficult to find assets, which are free from any risk. As a rule, the state securities are considered to be risk-free. However, even in this case, the risk exists, though, probably, it is considerably smaller in comparison with other financial tools.

In this case, we shall consider that there is an assets that is described by a certain rate of profitability and zero risk. In our diagram, in coordinates of risk and income, this is the point O that lays on the axis of the average income (fig. 18). We shall call the rate r_0 as the risk-free rate of profitability.

Combination the risky and risk-free investments

If there exists the risk-free rate of profitability, the investor faces the task of distributing his investments among some risky asset (a portfolio of assets) with the expected income $\bar{r}_p$ and square of PRC $\dot{r}_p^2$ and the risk-free investments of means with the rate $\bar{r}_0$, under $\bar{r}_p > \bar{r}_0$ (fig. 18). Let x_p is the share of wealth invested into risky business, x_0 - share invested at the risk-free rate.

The segment OA contains combinations of t risk and profitability attainable by investments into risky portfolio (point A) and risk-free asset (point O). If there are unlimited possibilities of the risk-free crediting, then all possible combinations of profitability and risk lay on the ray OAA$^{/}$.

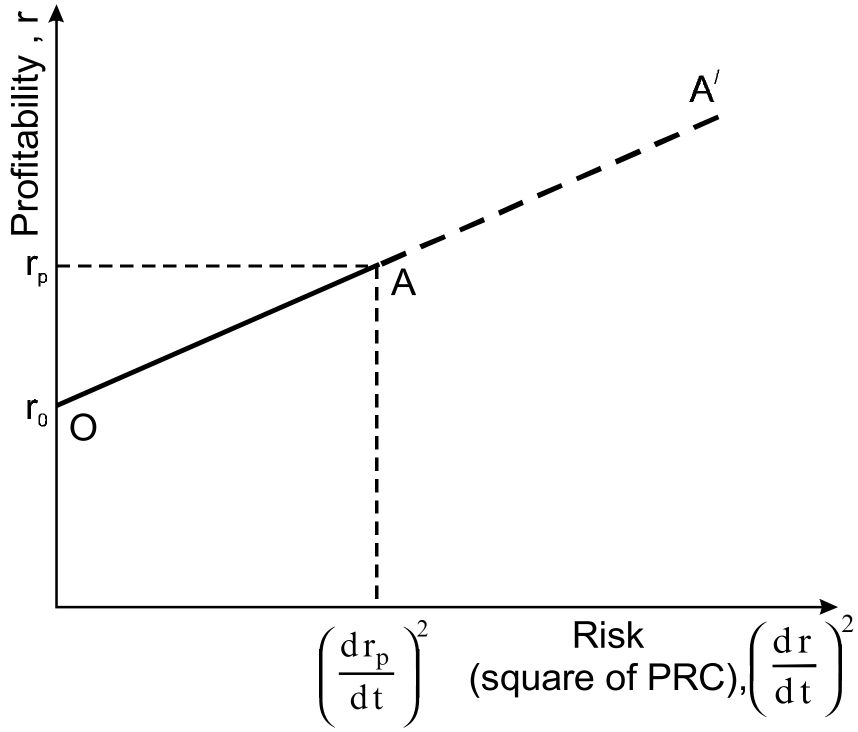


Fig. 18. Combination of risky and risk-free investments in the portfolio

Let's designate the profitability of the common portfolio as $\bar{r}'_p$:

$$\bar{r}'_p = x_p \bar{r}_p + x_0 \bar{r}_0. \tag{2.25}$$

The risk (square of PRC) of the common portfolio r'_p will be equal (taking into account that the point of time derivative from the constant is equal to zero)

$$\dot{r}_p^2 = x_p^2 \dot{r}_p^2 + x_0^2 \cdot 0 = \left| x_p \dot{r}_p \right|^2. \tag{2.26}$$

The equations (2.25) and (2.26) at $x_p \geq 0$, $x_0 \geq 0$, $x_p + x_0 = 1$ it parametrically set the segment OA in figure 18. If $x_p = 0$, then $x_0 = 1$ and parameters of the portfolio correspond to the point O. On the contrary, at $x_p = 1$ and $x_0 = 0$ all means are routed to risky investments, and the portfolio is determined by the point A. All possible intermediate values, when part of investments is risky, and part is risk-free, lay on the segment OA.

Risk-free borrowing

Previously, we spoke only about risk-free investments. We shall presume now that there is a possibility of not only investments, but also borrowings under the rate $\bar{r}_0$.

In other words, the investor may take credit under the rate $\bar{r}_0$ and invest these means into the risky portfolio. According to the above terminology, risk-free crediting is similar "short sale" of the risk-free asset. Using our designations, it is possible there is get into the situation, when $x_0 < 1$ and $x_p > 1$ (it is natural that budget limitation $x_p+x^0=1$ sttill exists). In this case, possible combinations of risk and income are also determined by equations (2.25) and (2.26), but these possibilities are not limited to the segment OA, and are extend on the whole ray OAA$^/$(combinations of risk and income, which are achievable through crediting, are displayed in figure 18 by the dashed line).

2.8. Risk-free rate and the effective set

Now we shall try to connect, on one hand, possibilities of risky investments that are determined by the set of admissible portfolios and by the set of appropriate admissible combinations of risk and income, and on the other, investments into risk-free assets. We shall consider, what will be the behaviour of the rational, not inclined to risk investor, when he simultaneously may form the portfolio from risky and risk-free assets. The picture now is essentially changed. We shall consider figure 19. Let risk-free rate of profitability be equal to $\bar{r}_0$ and the admissible set of the risky portfolios is limited to the curve EE$^/$.

The point M that lays on the tangent built from the point Of to the boundary of EE$^/$ effectiveness corresponds to the market portfolio. If there exists a risk-free rate of profitability, then investments into any other risky portfolio, which is different from the market one (for example, A and B), are ineffective.

The task of the investor now can be divided into two subtasks: first, it is necessary to choose the risky portfolio from the set of possible ones, second, to distribute means between risk-free investments and the risky portfolio. Which of the available portfolios will be chosen? We suppose that the rational investor always tries to choose an effective portfolio, in other words, the one with the average profitability and the risk laying on the boundary of the admissible set $EE^{/}$. Let's compare two portfolios, for example, A and B in figure 19. Admissible combinations of risk and income at various combinations of risk-free and risky investments, when choosing the portfolio A, are depicted by the ray OA, when choosing the portfolio B – by the ray OB. It is obvious that in case of the portfolio B, we, at any decision, receive the greater average returns at the same degree of risk. By comparing similarly portfolios M and B, we see that, in its turn, the portfolio M is better than the portfolio B.

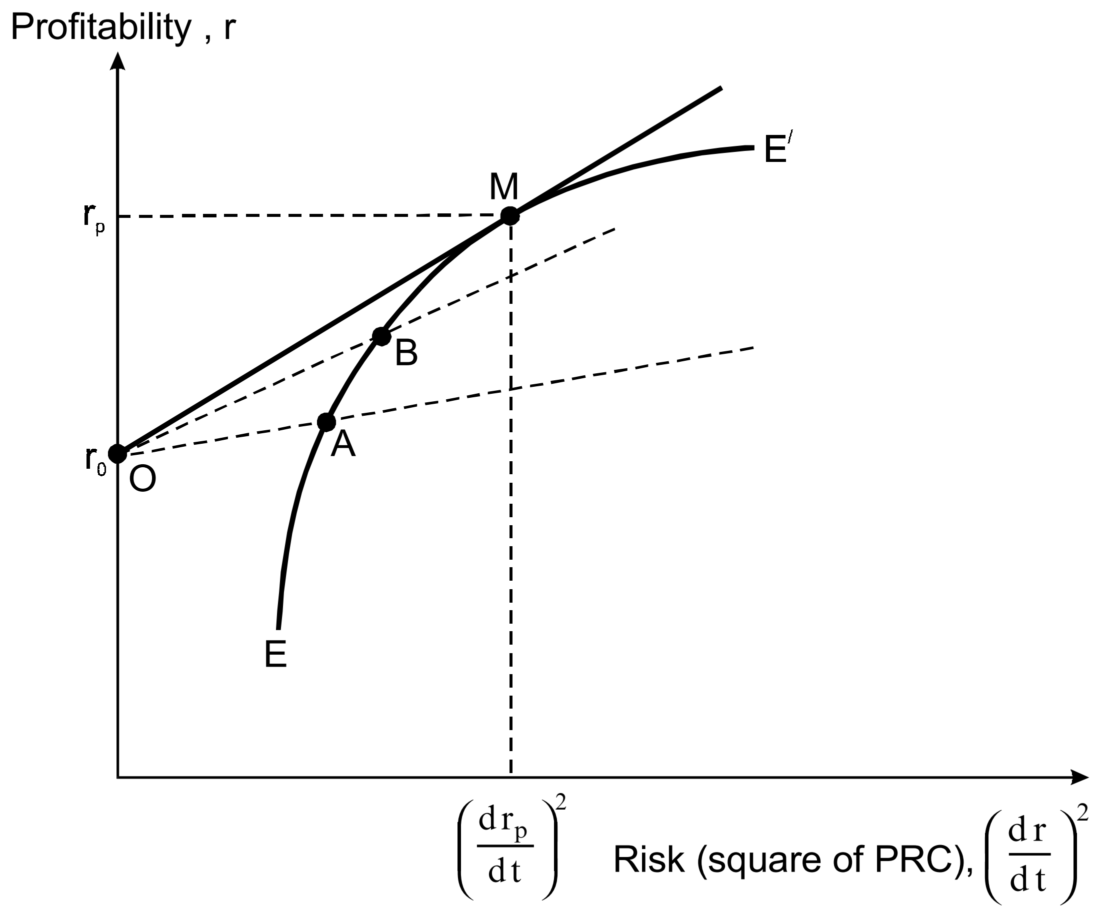


Fig. 19. Graphical determination of the portfolio with minimum risk

BThe conclusion is obvious: if there is the possibility of risk-free investments, then the best for any investor, who is not inclined to risk, will be that portfolio of risky assets, which corresponds to the point of contact of the ray drawn from the point O $\left(\bar{r}=\bar{r}_0, \dot{r}^2=0\right)$ to the boundary of effectiveness.

Thus, all investors will strive to invest their means into the portfolio

M. The only difference will be the proportions of the investors' distribution of their wealth between the risky and risk-free investments (in other words, between the portfolio M and the risk-free asset). Referring to a relatively more conservative investor (with the greater degree of disinclination to risk), he will choose the decision that is closer to the point O. The aggressive investor (less not inclined to risk) may decide to invest his means into the portfolio M not only at the expense of own means, but also at the expense of crediting under the rate $\bar{r}_0$ (fig. 20).

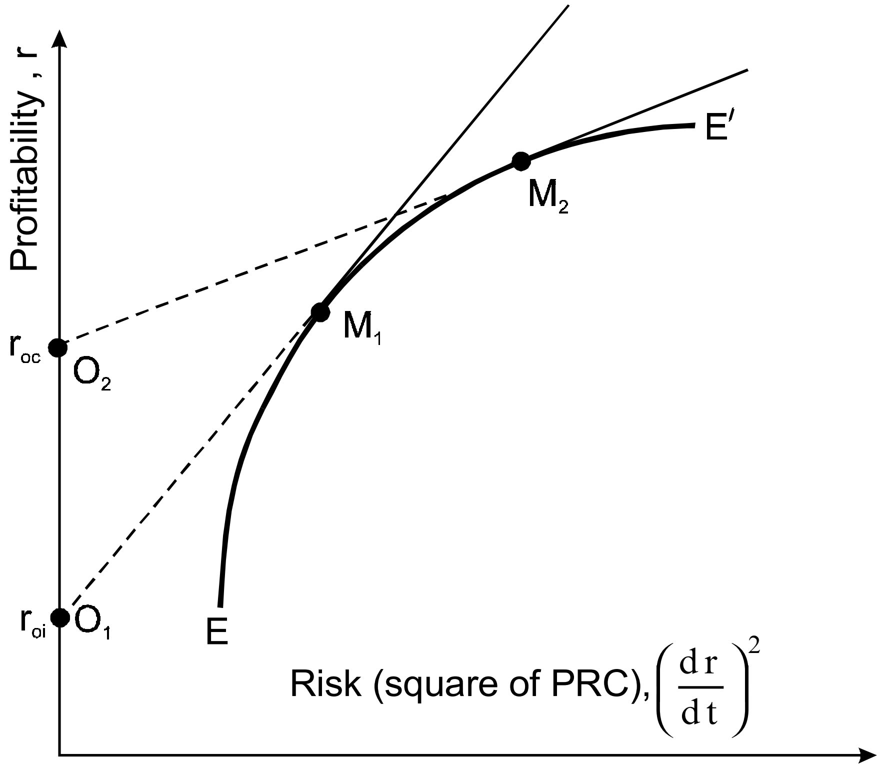


Fig. 20. "Profitability risk" dependence
for the "aggressive" and "conservative" investors

The boundary of effectiveness under the risk-free rate will be no more the curve EE/ that describes the possibilities of risky investments, but the right line (the ray is a more exact wording) that is the tangent to the admissible set drawn from the point O (line OM in figure 19).

2.9. Differences in risk-free rates

of crediting and investments

Let's consider now, how our conclusions will change, if we shall weaken some of our suppositions and make our model more real. Supposition about existence of the uniform risk-free rate for both the credits and investments is, naturally, a great simplification. In reality, we, as the rule, observed that the rates of profitability, which can be used by us for invest of our means, differ from the rates of crediting. And very frequently, the rate of profitability of investments for a separate investor is less than the rates at which he may borrow means. We shall designate $\bar{r}_{0c}$ - as the risk-free rate for loans, $\bar{r}_{0i}$ - risk-free rate of profitability for investments, under condition that $\bar{r}_{0c} > \bar{r}_{0i}$.

If rates for the risk-free crediting and investments are different, the form of the effectiveness boundary is changing, and there exists not the only optimal portfolio for all investors. The set of market portfolios is characterized by area M_1M_2 of the boundary MM'.

The effective set, thus, will change essentially. Let, as before, EE' is the boundary of the admissible set of combinations of risk and profitability for the risky assets. In case of combination of the risky and risk-free investments, the accessible combinations of risk and profitability lay on the segment O_1M_1, where M_1 is the point of contact of the ray drawn from the point O_1 (value $\bar{r}_{0i}$) to the curve of EE'. If the investor strives to get the greater income, than it can be provided by the portfolio M_1, and, accordingly, he is ready to run the greater risk, his choice may be the investments into onle one risky portfolios, the risk of profitability of which lays in the area M_1M_2 of the curve of EE'. At last, even more aggressive investor will choose the decision, at which the investments will done into the portfolio M_2 at the expense of crediting at the rate $\bar{r}_{0c}$. Possible combinations of risk and income in the latter case lay in the area of the tangent drawn from the point O to the curve EE', starting from the point M_2 (fig. 20).

Thus, if the rates of profitability of risk-free investments and credits differ, the set of effectiveness consists of three areas, and the portfolio of risky assets will not be the same. Depending on the degree of disinclination to risk, investors will choose different portfolios of risky assets, the risk and profitability of which lay in area M_1M_2 of the curve EE'.

2.10. Updated Markovitz's model

The model of the investor's behaviour, according to which the investments are evaluated exclusively by two parameters - expected profitability and risk that is measured as the square of PRC, allows to lay down the uniform rule of the portfolio forming, which is followed by all investors: independent of the individual preferences, all investors try to form the effective portfolio: the one, which ensures the minimum degree of risk for the chosen level of profitability, or, and it is the same, the maximum expected income at the set degree of risk. This approach and the task of choice of the effective portfolio will be called by us as the updated Markovitz's model.

Let, as before, there exists *n*-assets, each of which ensures an aleatory variable of profitability ξ_i (i=1, ..., n), $\bar{r}_i$ - expected (average) profitability of *i*-asset (expectation of the aleatory variable ξ_i):

$$\bar{r}_i = E\xi_i,$$

$$(\dot{r}_i, \dot{r}_j) = \|\dot{r}_i\| \cdot \|\dot{r}_j\| \cos\angle(\dot{r}_i, \dot{r}_j),$$

where $(\dot{r}_i, \dot{r}_j)$ - scalar product of PRC *i*- and *j*-assets;

$\cos\angle(\dot{r}_i, \dot{r}_j)$ - cosine of the angle between the averaged vectors of rates of change of profitability of two assets *i* and *j*.

The updated Markovitz's model can be laid down as follows: it is necessary to find such proportions of distribution of means between available assets $x_1, x_2..., x_n$ (where, x_i is the share of means invested into *i*-asset) that the risk of the portfolio $\dot{r}_p^2$ at this level of profitability $\bar{r}_p$ is minimum. Mathematically, the model can be laid down as follows:

to find

$$\min_{x_1, x_2, ..., x_n} \left\{\dot{r}_p^2\right\} = \min_{x_1, x_2, ..., x_n} \left\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \dot{r}_i \dot{r}_j\right\}, \tag{2.27}$$

at limitations

$$\sum_{i=1}^{n} x_i = 1, \quad \sum_{i=1}^{n} \bar{r}_p x_i = \bar{r}_p. \tag{2.28}$$

In the given formulation of the model $\bar{r}_p$ - is the set level of average profitability

$$(\dot{r}_i \cdot \dot{r}_j) = \begin{cases} \dot{r}_i^2, & i = j \\ \dot{r}_i \cdot \dot{r}_j, & i \neq j \end{cases}$$

(according to accepted designations $(\dot{r}_i \dot{r}_i) = \dot{r}_i^2$).

The model can be written down in the matrix form, where x is the vector of means distribution between the risky assets: $x=\{x_i\}_{i=1, ..., n}$; $\bar{r}$ - vector of profitability of assets, V - matrix of the scalar products of PRC (the square matrix consisting of values $(\dot{r}_i \dot{r}_j)$, i=1, ..., n; j = 1, ..., n). Then, it is necessary to find

$$\min_x \{x^T V x\}, \tag{2.29}$$

at limitations

$$x^T e = 1, \quad x^T \bar{r} = \bar{r}_p, \tag{2.30}$$

where e - unit vector:

$$e = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix},$$

^{T} - sign of the vector transposition.

For the model (2.29), (2.30) it is easy to find an analytical decision

$$x^* = \frac{1}{2} V^{-1} (\varphi \bar{r} + \psi e),$$

where φ and ψ - factors of Lagrange's limitations (2.30).

If there exists the risk-free asset, the model can be written down as:

$$\min_x \{x^T V x\}, \tag{2.31}$$

$$x^T \bar{r} + x_0 \bar{r}_0 = \bar{r}_p, \tag{2.32}$$

$$x^T e + x_0 = 1, \tag{2.33}$$

where $\bar{r}_0$ - risk-free rate,

x_0 - share of wealth invested into the risk-free asset.

It is possible to get rid of the limitation (2.32) by replacing:

$$x_0 = 1 - x^T e. \qquad (2.34)$$

Then, limitation (2.33) will look as:

$$x^T(\bar{r} - \bar{r}_0 e) = \bar{r}_p - \bar{r}_0, \qquad (2.35)$$

and the solution of the task (2.31) - (2.33) can be written down as:

$$x^* = \frac{1}{2} V^{-1}(\bar{r} - \bar{r}_0 e)\varphi,$$

where φ - factor of the Lagrange's limitation (2.35).

2.11. Updated model of financial assets estimation

The problem of investments and financial titles estimation, which "promise" the not guaranteed recurrent flow in future by investigating the prices of other financial titles, which circulate in the perfectly functioning market of capital, was considered in the theory of arbitration. The most important problem of his analysis is, whether the economic estimation of the financial title is reasonable in relation to other titles circulating in the market.

However, the whole market of capital is not considered in the theory of arbitration. It is possible to say that this theory takes into account only a relative small number of financial titles and investigates only prices equilibrium in reference to each other. Its opposite model is the CAPM (Capital Asset Pricing Model) that was developed in the 60th of XX century by J. Lintner, Jean Mossin and William F.Sharp. CAPM has allowed to determine the price of capital markets requirements – requirements to the unsecured future recurrent flow.

The major basis for the CAPM was the theory of portfolio choice, which in the 50th of XX century was created by Harry M. Markovits. Within this concept, investors have the possibility of choosing risky securities (share) in such a way that for this level of risk the expected income is maximized.

The main idea was to realize the risk diversification, i.e., to choose the securities, which are different in the dynamics of their rates. It is known that "it is impossible to put all eggs into one basket. If this thesis is applied to shares, this would mean that it is very risky to spend all your property on purchase of the only title. As, if the future of this industry will bee unsuccessful, losses are inevitable.

The empirical researches in early 70th of the last century, which devoted mainly to the USA markets of capital, have shown that CAPM has a number of drawbacks, the main of which are:

1) Beta is not the only risk factor that explains the profitability of risky financial titles;
2) Checks of quotations show that the linear dependence does not always exist between the profitability and Beta.

In 1977, Richard Roll published the article, in which he doubted the fact that the link "profitability – risk" that is defended within the CAPM, can be verified empirically.

Let's construct the theory of financial assets estimation by using the principle of the investor' s optimal behaviour (2.3).

As an alternative, the authors of CAPM offer the one-period model, which is discussed below. At the point of time t = 0, all decisions are made, at the point of time t = 1, the "harvest" of these decisions is summed up and consumed.

Let presume that, in economics, there exists the *i*-number of individuals, which should make decisions on their plans of consumption at the point of time t = 0. Each of them has an initial store of consumer benefits and financial titles, which can be specified as follows.

$\overline{C}_0^i \geq 0$ is the initial store of consumer benefits in possession of the *i*-individual at the point of time $t = 0, i = 1..., I$. The amount of all consumer benefits, which exist in economy as the whole, are represented by the today's supply of consumer benefits and may be expressed by the sum: $\sum_{i=1}^{I} \overline{C}_0^i$.

We understand that none of the individuals is obliged to consume all initial store of consumer benefits. He may purchase additional consumer benefits or sell them. The amount of consumer benefitsб which e *i*-individual at the point of time t = 0 can actually consume is designated by us as C_0^i, so total demand of consumer benefits can be laid down as: $\sum_{i=1}^{I} C_0^i$.

At the point of time t = 0, trading in consumer benefits is realized at the price ψ_0 and at the point of time t = 1 – at the price ψ_1. We presume that there is no inflation and, therefore, we may not take into account the index of time. So, we write only ψ.

Individual at the point of time t = 0 own not only consumer benefits, but also financial assets in the form of securities. As a whole, there exist J + 1 of different financial titles with properties, which are considered in details below.

A primitive financial title is risk-free securities that promises that its holder, at the point of time t = 1 the guaranteed income in the amount of $r_0 = 1$ \$ and at the point of time t = 0 is sold and purchased at the price of $p(r_0) = 1/(1+\alpha_f)$, where α_f — is the risk-free interest rate. The number of risk-free financial titles in possession of the *i*-participant as the initial store is designated as $\overline{n}_0^i$.

While considering all other securities, we consider risky financial titles, which are owned by holders, which have payments requests of the not yet determined value: *j*-title with j = 1,..., J "promises" at the point of time t =1 payments equal to r_{j1}, ..., r_{jS} depending on the future situation. If at the point of time t = 1, *s*-situation occurs, the individual that possesses the unit of *j*-title receives the money flow equal to r_{jS}. Expectation of the money flow of the *j*-financial title shall be designated by us as $E(\tilde{r}_j)$, and the scalar product of rates of change of profitability (PRC) of two securities, as *j* and *k*, accordingly, $(\dot{r}_j, \dot{r}_k)$.

Identical price of this financial title for all participants of the market is equal to $p(\tilde{r}_j)$, and $\overline{n}_j^i$ means the amount of *j*-financial title that is in possession of the *i*-participant of the market at the point of time t = 0. We do not put a dash over the symbol, if want to designate the amount of the *j*-financial title, which is in possession of the individual at the point of time t = 1.

If we use the represented symbols, **the store of consumer benefits of the** *i*-individual at the point of time t = 0 is equal to:

$$\overline{C}_0^i,$$

while his store of financial assets is equal to:

$$\overline{n}_0^i \cdot \frac{1}{1+\alpha_f} + \sum_{j=1}^{J} \overline{n}_j^i \, p(\tilde{r}_j)$$

To find the **total store of the individual**, it is necessary to decide, how it should be expressed - in monetary units or in units of consumer benefits. In the first case, we have:

$$\psi \bar{C}_0^i + \bar{n}_0^i \cdot \frac{1}{1+\alpha_f} + \sum_{j=1}^{J} \bar{n}_j^i \, p(\tilde{r}_j). \qquad (2.36)$$

The individual, who does not want any savings, may, at the point of time t = 0, after sale of all his financial titles, spend them on consumer benefits equal to the value of the expression (2.36). If costs of consumer benefits at the point of time t = 0 are less than this sum, then the individual is simultaneously the saver and investor. Savings are invested into financial titles. The individual with a rather low norm of time preferences will be not only the holder of the initial store of financial titles, but also he will expand this store at the expense of sale of the part of his store of consumer benefits to enable larger consumption in the future. On the contrary, individuals with the high norm of time preferences sell part of their store of financial titles to exchange their cost for the consumer benefits planned for current consumption.

Using the introduced symbols, the existing in the economy **j-type prepositions of risky financial titles** can be formalistically described. At their quantitative measurement, we receive the expression:

$$\sum_{i=1}^{I} \bar{n}_j^i,$$

and, at cost measurement, the received expression should be multiplied to the price $p(\tilde{r}_j)$

$$\sum_{i=1}^{I} \bar{n}_j^i \, p(\tilde{r}_j).$$

Dependence of the future consumption on the situation. The consumption level, which can be afforded by the individual at the point of time t = 1, depends on what financial title was in his possession at the point of time t = 0 and what situation will occur at the point of time t = 1. Financial titles of the investor bring, at the point of time t = 1, flows of payments that are dependent on the situation and the value equal to:

$$n_0^i r_0 + \sum_{j=1}^{J} n_j^i \, r_{js}.$$

It means, that the value of financial titles is not guaranteed until it is known, what *s* situation will occur at the point of time t = 1. The same is real for consumer benefits, which can be afforded by *i*-participant of the market based on the income from these financial titles. They are also dependent on the situation and equal to:

$$C_{1s}^{i} = \frac{n_0^i + \sum_{j=1}^{J} n_j^i r_{js}}{\psi}. \qquad (2.37)$$

(We assume that $r_0 = 1$).

Utility functions. Let's presume that all participant of the market want to realize their consumer plans which will help them to maximize their individual utility. We cannot say that the system of preferences of individuals are identical, however, they are based on one and the same principles.

The below model is based on the fact that behaviour of all economic subjects meets such axioms as:

Axiom of comparability. Evaluating alternatives z_1 and z_2, each decision making person will manage to choose some alternative or will accept both alternatives as equivalent, therefore, either $z_1 \geq z_2$ or $z_1 \sim z_2$, or $z_1 \leq z_2$. Other variant of decision-making is excluded.

Axiom of transitivity. If variant z_1 for the individual is, as minimum, the same as z_2, and simultaneously, the individual supposes that z_2 is not worse, than z_3, then this individual should think that z_1 is at least as good as z_3.

Axiom of continuity. If $z_1 > z_2 > z_3$, then for each alternative there exists one (individual) probability $q \in (0, 1)$ at which $z_2 \sim [z_1, z_3: q, 1 - q]$.

Axiom of limitation. Among possible consequences of the accepted decision for each person, there exists the best and the worst result, $\bar{z}$ and $\underline{z}$.

Axiom of prevalence. If someone has the relation of preferences $\bar{z} > \underline{z}$, then he should evaluate the lottery $[\bar{z}, \underline{z},: q_1, 1-q_1]$ higher, than the lottery $[\bar{z}, \underline{z},: q_2, 1-q_2]$, if $q_1 > q_2$, in other words, if more a more attractive alternative is offered with higher probability.

Axiom of independence. If there is the relation of preference $z_1 \geq z_2$, then this the relation should related to two lotteries, which differ by only the fact that z_1 substitutes z_2. It means that $[z_1, z_3: q, 1 - q] \geq [z_2, z_3: q, 1 - q]$.

Monotonicity. We presume that the needs of each managing subjects are not satisfied. For this reason, for all of them, at $\delta > 0$ the following is true:

$$(C_0 + \delta, C_1) > (C_0, C_1) \text{ и } (C_0, C_1 + \delta) > (C_0, C_1).$$

Upward convexity of the utility function. This assumption together with the appropriate premise of budget limitations reduces to the uniqueness of the optimal consumer plan of the investor. To achieve the same purpose and under uncertainty, we shall presume that the utility function is the upward convex one. With the help of this premise, it is possible to achieve the uniqueness of optimal decisions. Then, we call the utility function as the univalently upward convex function, if for all *x* at $0 < x < 1$ the following is true:

$$U(xC_0 + (1 - x)C_1) > x\, U(C_0) + (1 - x)\, U(C_1) \; \forall \; C_0, C_1.$$

The upward convex utility functions have two important properties. On the one hand, upward convex utility function always means strict aversion of risk, as it is already clear to us, but, on the other hand, such function also presupposes the decreasing marginal utility, in the same way as downward also curves of equal probability.

Regularity. For simplification purposes, we presume that functions of utility of all persons, who make decisions, are continuous and double differentiated. It is further presumed that it is possible to calculate expectations of utility, and that arguments of the operator of expectation are double differentiated. This assumption should lead to fact that optimal consumer plans can be determined with the help of differential calculus.

Specification of the utility function. As we were basing on the fact that consumption at the point of time $t = 0$ is guaranteed, while consumption at the point of time $t = 1$ is not guaranteed, then we need the inter-time function of utility, which is for the decision making *i*-person may be laid down as follows:

$$U = U(r_i). \tag{2.38}$$

Then, we presume that all persons make decisions referring to the unsecured consumption based on expectations of profitability of securities and profitability rate of change, in other words, on the basis of the principle (2.3).

Uniform expectations

All participant of the market are unanimous in what payments "are promised" by the *j*-title to its holder, if the future s-situation takes place. So, there is the general opinion that such situations, at the point of time t = 1, will not univalently happen.

In connection with homogeneity, we presume that probabilities of possible situation existence are equally evaluated by all participants of the market. This, as opposed to the theory of arbitration, means that all managing subjects are characterized by identical average expectations and scalar products of PRC. If we used this strong supposition, it makes sense to think of, whether it is true or not. It will be such in case, if information that is essential for estimation of financial titles, is spreading quickly in the market and the market participant (for which it is important) have approximately similar mental abilities of processing this information.

Suppositions about the market

Let's also presume that the market process is functioning uninterrupted and that markets are competitive. Therefore, financial titles and consumer benefits can be divided indefinitely. There are no transaction costs, no taxes, no limitations of access to the market. Sales without cover are admissible. Because of the atomistic structure of the market, all it participants adapt themselves to changes in market conditions by quantitative changes and accept the market prices as the set ones. At last, it is presumed that markets in equilibrium suggest no possibilities for arbitration.

The consumer plan of each individual is guaranteed at the point of time t = 0 and is not guaranteed at the point of time t = 1. While choosing the plan of consumption, it is necessary to take into account budget limitations for the point of time t = 0 and t = 1. If we used already introduced symbols and disengage ourselves from the i-index that designates a certain investor, we can lay down the problem of decision-making in the form of (2.38) with additional conditions:

$$\psi C_0 + \frac{n_0}{1+\alpha_f} + \sum_{j=1}^{J} n_j p(\bar{r}_j) = \psi \bar{C}_0 + \frac{\bar{n}_0}{1+\alpha_f} + \sum_{j=1}^{J} \bar{n}_j p(\bar{r}_j), \tag{2.39}$$

$$\psi C_{1s} = n_0 + \sum_{j=1}^{J} n_j r_{js}. \tag{2.40}$$

The investor makes decisions with reference to the today's consumption C_0 and the set of financial titles n_0, n_1..., n_J. Using these variables, according to (2.37), he also determines consumption that is dependent on the situation in the point of time t = 1. Condition (2.38) means that the budget at the point of time t = 0 should be in equilibrium. Financial titles purchase costs and costs of consumer benefits should correspond to the property, which is owned by the individual as the initial store. Condition (2.39) reduces to equivalence between the situation dependent costs on consumption at the point of time t = 1 and uncertain recurrent flows generated by the store of financial titles.

With the purposes of simplification, we use the price of consumer benefits as the unit of measurements. We define $\psi = 1$ and the above problem of optimization may written down in the form (2.38) under additional condition:

$$C_0 + \frac{n_0}{1+\alpha_f} + \sum_{j=1}^{J} n_j p(\bar{r}_j) = \bar{C}_0 + \frac{\bar{n}_0}{1+\alpha_f} + \sum_{j=1}^{J} \bar{n}_j p(\bar{r}_j), \tag{2.41}$$

$$C_{1s} = n_0 + \sum_{j=1}^{J} n_j r_{js}. \tag{2.42}$$

<u>Application method of the Lagrange's factors</u>. Applying the method of the Lagrange's factors for solving this optimization task, we resort to further simplification. Let's lay down formulas of expectation and the scalar product of profitability of rates of change using (2.42), and we shall substitute the received expressions into the Lagrange' s function. It pays because it is necessary to substitute only condition (2.41) into the Lagrange's function. Future consumption is the aleatory variable, which can be represented in the following form according to (2.42)

$$\overline{C}_1 = n_0 + \sum_{j=1}^{J} n_j r_{js}.$$

In this case, rules of the linear combination are true for expectation and the scalar product and we receive:

$$E(\overline{C}_1) = n_0 + \sum_{j=1}^{J} n_j E(\overline{r}_j). \quad (2.43)$$

$$\dot{r}^2(\overline{C}_1) = \sum_{j=1}^{J}\sum_{k=1}^{J} n_j n_k (\dot{r}_j, \dot{r}_k). \quad (2.44)$$

Then the Lagrange' s function has the following form:

$$L' = L\left(C_0, E(\overline{C}_1), \dot{r}^2(\overline{C}_1)\right) +$$

$$+k\left(C_0 + \frac{n_0}{1+\alpha_f} + \sum_{j=1}^{J} n_j p(\overline{r}_j) - \overline{C}_0 - \frac{\overline{n}_0}{1+\alpha_f} - \sum_{j=1}^{J} \overline{n}_j p(\overline{r}_j)\right), \quad (2.44')$$

and $E(\overline{C}_1)$ and $\dot{r}^2(\overline{C}_1)$ are represented in this function in the form (2.43) and (2.44).

Condition of the first order. If to take partial derivatives according to the C_0, n_0, ..., n_J and k and put them to zero, the following set of equations will be received:

$$\frac{\partial L'}{\partial C_0} = \frac{\partial L}{\partial C_0} + k = 0, \quad (2.45)$$

$$\frac{\partial L'}{\partial n_0} = \frac{\partial L}{\partial E(\overline{C}_1)} + \frac{k}{1+\alpha_f} = 0, \quad (2.46)$$

$$\frac{\partial L'}{\partial n_j} = \frac{\partial L}{\partial E(\overline{C}_1)} \cdot E(r_j) + \frac{\partial L}{\partial \dot{r}^2(\overline{C}_1)} \cdot 2\sum_{k=1}^{J} n_k (\dot{r}_j, \dot{r}_k) +$$
$$+ kp(\overline{r}_j) = 0, \quad j = 1, \dots, J, \quad (2.47)$$

$$\frac{\partial L'}{\partial k} = \left(C_0 + \frac{n_0}{1+\alpha_f} + \sum_{j=1}^{J} n_j p(\overline{r}_j) - \overline{C}_0 - \frac{\overline{n}_0}{1+\alpha_f} - \sum_{j=1}^{J} \overline{n}_j p(\overline{r}_j)\right) = 0. \quad (2.48)$$

Probably, it is useful to give some additional information referring to the way of receiving expressions (2.46) and (2.47). If we want to differentiate the utility function according to n_0, it is necessary to explain that n_0 is the component of expectation of future money flows $E(\overline{C}_1)$, see (2.43). For this reason, at first, we take the derivative according to $E(\overline{C}_1)$, and after that it is multiplied by the derivative of expectation according to n_0. Their product is equal to one.

The same procedure should be used, if it is necessary to differentiate the Lagrange' s function according to n_j. Here, it is necessary to take into account the fact that n_j is the component of both the expectation of future money flows, and the rates of change of profitability of these flows, see (2.43) and (2.44). The derivative of expectation according to n_j is equal to $E(\bar{r}_j)$. More attention will be required by the derivative r_j according to n_j. to illustrate the result received due to the formula (2.47), see tab. 9.

Table 9. Matrix of the scalar product of profitability of rates of change

	1	…	j	…	J
1			$n_1 n_j(\dot{r}_1, \dot{r}_j)$		
…			…		
j	$n_j n_1(\dot{r}_j, \dot{r}_1)$	…	$n_j n_j(\dot{r}_j, \dot{r}_j)$	…	$n_j n_J(\dot{r}_j, \dot{r}_J)$
…			…		
J			$n_J n_j(\dot{r}_J, \dot{r}_j)$		

In table 9, matrix consists of $n_j n_k(\dot{r}_j, \dot{r}_k)$- type elements. Basing on (2.44) it follows that, if we summarize all elements of matrix by lines and columns, we will received exactly the $\dot{r}^2(\overline{C}_1)$. It is apparent in the table that we have singled out only those elements, which contain n_j, as all other elements during derivation according to the n_j turn to zero.

In the area of intersection of j-lines with j-m column there is an element that can be also written down in the form $n_j^2(\dot{r}_j, \dot{r}_j)$, while all other elements of interest to us are dubbed, if we to take into account that $(\dot{r}_j, \dot{r}_k) = (\dot{r}_k, \dot{r}_j)$. As a result, we can write down the sum of matrix elements fit for differentiation in the form of the auxiliary quantity:

$$A = n_j^2(\dot{r}_j, \dot{r}_k) + 2\sum_{\substack{k=1 \\ k\neq j}}^{J} n_k n_j (\dot{r}_j, \dot{r}_k).$$

If we differentiate these auxiliary quantities according to n_j, we shall receive:

$$\frac{\partial A}{\partial n_j} = 2n_j(\dot{r}_j, \dot{r}_j) + 2\sum_{\substack{k=1 \\ k\neq j}}^{J} n_k n_j (\dot{r}_j, \dot{r}_k) =$$

$$= 2\sum_{k=1}^{J} n_k (\dot{r}_j, \dot{r}_k).$$

Thus, we have completed explanation of formation of the partial derivative of the Lagrange's function.

2.12. Analysis of the first order conditions

First order conditions (2.45) and (2.48) completely describe the optimal actions of the i-individual. In the state of market equilibrium, actions of all individuals should correspond to each other. This happens only when the demand is equal to the supply, and, therefore, for financial titles, the following is true:

$$\sum_{i=1}^{I} n_j^i = \sum_{i=1}^{I} \bar{n}_j^i \text{ for } \forall \text{ j.} \tag{2.49}$$

Naturally, the prices, which are formed in such a way so that to attain the market equilibrium, are the prices $p(r_0), p(\bar{r}_1), ..., p(\bar{r}_J)$.

2.12.1. Risk-free interest rate and time preference

If to express k from (2.45) and then to substitute the result in (2.46), then after a minor conversion, we shall receive:

$$\frac{\partial L / \partial C_0}{\partial L / \partial E(\overline{C}_1)} = 1 + \alpha_f \; -$$

the result that in a very similar form can be received under condition of expectations coming true. In optimum, the relation of the marginal utility C_0 to the marginal utility of the expected consumption C_1 is equal to the area of values $(1+\alpha_f)$. Thus, the inter-time compromise choice "consumption-investments" is done by the investor under condition of unsecured recurrent flows. Refusal from consumption at the point of time t = 0 continues until the personal norm of time preference does not coincide with risk-free interest rate.

2.12.2. Price of the title with the unsecured income

Now we'd like to clarify, what are the factors that determine the price of the financial title with the unsecured (risky) income, if the investor has optimally made the compromise choice "consumption-investments". For this purpose, we concentrate our attention on the expression (2.47.) We shall transform it in such a way so that to form the relation, which will give the answer to this problem.

If we express k from (2.46) and insert it into (2.47), then, after elementary transformation we shall receive:

$$E(\bar{r}_j)-(1+\alpha_f)p(\bar{r}_j)=-2\frac{\partial L/\partial \dot{r}^2(\bar{C}_1)}{\partial L/\partial E(\bar{C}_1)}\cdot\sum_{k=1}^{J} n_k(\dot{r}_j,\dot{r}_k)$$

or after return to the expression that describes each separate investor

$$E(\bar{r}_j)-(1+\alpha_f)p(\bar{r}_j)=-2\frac{\partial L^i/\partial \dot{r}^2(\bar{C}_1)}{\partial L^i/\partial E(\bar{C}_1)}\cdot\sum_{k=1}^{J} n_k^i(\dot{r}_j,\dot{r}_k). \qquad (2.50)$$

Having designated the fraction $-2\frac{\partial L^i/\partial \dot{r}^2(\bar{C}_1)}{\partial L^i/\partial E(\bar{C}_1)}$ in this formula through h^i, we shall receive:

$$\frac{1}{h_i}=\frac{\sum_{k=1}^{J} n_k^i(\dot{r}_j,\dot{r}_k)}{E(\bar{r}_j)-(1+\alpha_f)p(\bar{r}_j)}.$$

After that, we shall sum all investors and receive:

$$\frac{1}{H}=\sum_{i=1}^{I}\frac{1}{h^i}=\frac{\sum_{i=1}^{I}\sum_{k=1}^{J} n_k^i(\dot{r}_j,\dot{r}_k)}{E(\bar{r}_j)-(1+\alpha_f)p(\bar{r}_j)}.$$

(2.51)

Let's concentrate our attention on the numerator of the right member. According to the rules of computation of the scalar product, we may also write it down as follows:

$$\sum_{i=1}^{I}\sum_{k=1}^{J} n_k^i\left(\dot{r}_j, \dot{r}_k\right)=\left(\dot{r}_j, \frac{\partial}{\partial t}\sum_{i=1}^{I}\sum_{k=1}^{J} n_k^i r_k\right).$$

If now this formula is substituted into (2.31) and H is expressed from it, we shall receive:

$$H=\frac{E\left(\bar{r}_j\right)-\left(1+\alpha_f\right)p\left(\bar{r}_j\right)}{\left(\dot{r}_j, \frac{\partial}{\partial t}\sum_{i=1}^{I}\sum_{k=1}^{J} n_k^i r_k\right)}. \quad (2.52)$$

This expression can be simplified afterwards.

Market portfolio. For this purpose, let's pay our attention to the market portfolio:

$$\bar{r}_m=\sum_{i=1}^{I}\sum_{k=1}^{J} n_k^i \bar{r}_k. \quad (2.53)$$

This formula describes the unsecured money flows, which will be received by all managing subjects as a whole, if they show the optimal demand for separate risky financial titles. Permutation of factors in the right member and change of summing priority will reduce to:

$$\bar{r}_m=\sum_{k=1}^{J}\sum_{i=1}^{I} \bar{r}_k n_k^i=\sum_{k=1}^{J} \bar{r}_k \sum_{i=1}^{I} n_k^i. \quad (2.52a)$$

In equilibrium of the sum of risky securities financial titles, which are of demand, should exactly coincide with the sum of the offered financial titles. Therefore, if we use (2.49) for money flows, the market portfolio may be written down also as:

$$\bar{r}_m=\sum_{i=1}^{I}\sum_{k=1}^{J} \bar{n}_k^i \bar{r}_k.$$

Here, important is the indication that the recurrent flows of the market portfolio for all investors are identical and exogenously set. Therefore, we may disengage ourselves from the index i that denotes a certain investor. The same is true for auxiliary quantities H. Thus, (2.52) looks like:

$$H = \frac{E(\bar{r}_j) - (1+\alpha_f)p(\bar{r}_j)}{(\dot{r}_j, \dot{r}_k)}. \tag{2.53}$$

If we shall find from this formula the price of risky financial title and receive:

$$p(\bar{r}_j) = \frac{E(\bar{r}_j) - H(\dot{r}_j, \dot{r}_k)}{1+\alpha_f}. \tag{2.54}$$

Market profitability. It is necessary for us to describe the auxiliary quantity H even more precisely. For this purpose, we shall transform (2.53) and multiply both members by n^i_j. This will reduce us to:

$$n^i_j H \cdot (\dot{r}_j, \dot{r}_m) = n^i_j E(\bar{r}_j) - (1+\alpha_f) n^i_j p(\bar{r}_j).$$

If we sum all financial titles and investors, then, taking into account (2.53), we shall receive:

$$\sum_{i=1}^{I}\sum_{j=1}^{J} n^i_j H \cdot (\dot{r}_j, \dot{r}_m) = \sum_{i=1}^{I}\sum_{j=1}^{J} n^i_j E(\bar{r}_j) - (1+\alpha_f)\sum_{i=1}^{I}\sum_{j=1}^{J} n^i_j p(\bar{r}_j),$$

$$H \cdot \left(\dot{r}_m, \frac{\partial}{\partial t}\sum_{i=1}^{I}\sum_{j=1}^{J} n^i_j r_j\right) = E\left(\sum_{i=1}^{I}\sum_{j=1}^{J} n^i_j \bar{r}_j\right) - (1+\alpha_f)\sum_{i=1}^{I}\sum_{j=1}^{J} n^i_j p(\bar{r}_j),$$

$$H(\dot{r}_m, \dot{r}_m) = E(\bar{r}_m) - (1+\alpha_f)p(\bar{r}_m).$$

When determining the profitability of market securities with the unsecured income

$$\overline{\alpha}_m = \frac{\bar{r}_m}{p(\bar{r}_m)} - 1$$

we may further write down

$$H(\dot{\bar{r}}_m, \dot{\bar{r}}_m) = E(\bar{r}_m) - (1+\alpha_f) \cdot p(\bar{r}_m),$$

$$H \cdot p^2(\bar{r}_m) \cdot (\dot{\alpha}_m, \dot{\alpha}_m) = p(\bar{r}_m) \cdot (1 + E(\bar{\alpha}_m) - 1 - \alpha_f),$$

$$H \cdot p(\bar{r}_m) \cdot (\dot{\alpha}_m, \dot{\alpha}_m) = E(\bar{\alpha}_m) - \alpha_f,$$

$$H = \frac{E(\bar{\alpha}_m) - \alpha_f}{(\dot{\alpha}_m, \dot{\alpha}_m)} \cdot \frac{1}{p(\bar{r}_m)}. \tag{2.55}$$

Thus, we have managed to receive the representation of H capable of interpretation.

Deduction of the formula of price. To finish with mathematical calculations, we shall start with (2.54). When using the profitability of the market portfolio, this formula can be represented as follows:

$$p(\bar{r}_j) = \frac{E(\bar{r}_j) - H \cdot \left(\dot{r}_j, \left[\frac{\partial}{\partial t} p(r_m) \cdot (1 + \bar{\alpha}_m) \right] \right)}{1+\alpha_f}.$$

Finally, we receive the formula:

$$p(\bar{r}_j) = \frac{E(\bar{r}_j) - H \cdot p(\bar{r}_m)(\dot{r}_j, \dot{\alpha}_m)}{1+\alpha_f}. \tag{2.56}$$

Substitution (2.55) into this formula after the elementary transformation, will reduce it to the following result:

$$p(\bar{r}_j) = \frac{E(\bar{r}_j) - \frac{E(\bar{\alpha}_m) - \alpha_f}{(\dot{\alpha}_m)^2} \times (\dot{r}_j, \dot{\alpha}_m)}{1+\alpha_f}. \tag{2.57}$$

2.12.3. Price representation of VET

If we have the special task of finding the formula for estimation of risky investments, it would be reasonable to start with the formula that solves the problem under conditions of certainty, and on its basis to make appropriate changes. For this purpose, we shall recollect that with reference to conditions of certainty we have found the formula of price for the money flow that is generated by the financial title in one year,

$$p(r_j) = \frac{r_j}{1+\alpha_f}.$$

If the recurrent flows are not secured, then, the investor, who is not inclined to risk, will try to pay lower price. Risky money flows are less attractive, than secured.

The necessary adjustment of the formula of price could be carried out, in principle, by changing the numerator or the denominator of the right member. In the first case, the expected recurrent flows should be reduced to the value of the discount for the risk, i.e.:

$$p(\bar{r}_j) = \frac{E(\bar{r}_j) - \text{discount for the risk}}{1+\alpha_f}.$$

In this case, as the rate of discounting, we take, as before, the interest rate of risk-free financial titles, but prior to discounting, the expected recurrent flows should be decreased by the value of the "appropriate discounts for the risk". In the second case, the denominator should be changed, and the formula looks as follows:

$$p(\bar{r}_j) = \frac{E(\bar{r}_j)}{1+\alpha_f + \text{risk premium}}.$$

Thus, the expected money flow might be discounted not by means of the risk-free interest rate, but by means of the interest rate enlarged to the value of "appropriate risk premium". However, without the model of market capital we cannot make improvements referring to the risk discount and premium.

Adjustment of recurrent flows taking risk into consideration. To take into account the appropriate discount for the risk, referring to j-securities, it is necessary to the multiply the scalar product by the value that it is the same for all financial titles and therefore, we shall to call it as the market price of the risk

$$\text{Market price} \quad (\text{tau}) = \tau = \frac{E(\bar{\alpha}_m) - \alpha_f}{(\dot{\alpha}_m)^2} \tag{2.58}$$

at the typical for this title value of risk:

$$\text{Value of risk} = (\dot{r}_j, \dot{\alpha}_m). \tag{2.59}$$

It is noteworthy that the value of risk is measured not by the square of PRC, but by the scalar product. Therefore, the value of risk of a separate security is not taken into account. What is important is the "contribution" of the securities risk into the total portfolio of risky financial investments.

Substitution of (2.58) into (2.57) gives

$$p(\bar{r}_j) = \frac{E(\bar{r}_j) - \tau \cdot (\dot{r}_j, \dot{\alpha}_m)}{1 + \alpha_f}. \quad (2.60)$$

Here we may state: to find the price of the j-financial title, it is necessary to adjust the expected recurrent flows of these securities with an allowance for risk by their reduction by the product of the market price of risk and the scalar product between the recurrent flows of this title and the market profitability. The result, which can be interpreted as the risk-free equivalent for the expected recurrent flows of the j-title, should be discounted using the risk-free interest rate.

Adjustment of the interest rate with allowance for risk. To receive the formula of price on the basis of the risk adjusted rate of discounting, we shall concentrate our attention on the scalar product in (2.60). Profitability of the j-financial title can be represented as follows:

$$\bar{\alpha}_j = \frac{\bar{r}_j}{p(\bar{r}_j)} - 1. \quad (2.61)$$

Let's lay down the scalar product of PRC using the already familiar properties:

$$(\dot{\bar{r}}_j, \dot{\bar{\alpha}}_m) = \left(p(\dot{r}_j) \cdot \frac{\partial(1 + \dot{\alpha}_j)}{\partial t}, \alpha_m \right) =$$

$$= p(\bar{r}_j) \cdot \left(\frac{\partial(1 + \dot{\alpha}_j)}{\partial t}, \alpha_m \right) =$$

$$= p(\bar{r}_j) \cdot (\dot{\alpha}_j, \dot{\alpha}_m).$$

Substitution into (2.60) will lead us, after the elementary transformation, to the following formula:

$$p(\bar{r}_j) = \frac{E(\bar{r}_j)}{1 + \alpha_f + \tau \cdot (\dot{\alpha}_j, \dot{\alpha}_m)}. \qquad (2.62)$$

Therefore, the risk premium as the make-weight to the risk-free interest rate is equal to the product of the market price of risk (τ) and the value of risk:

$$\text{Value of risk} = (\dot{\alpha}_j, \dot{\alpha}_m). \qquad (2.63)$$

2.12.4. Representation of VET in terms of profitability

VET is expresses by the formula of profitability that can be expected by the investor in the state of equilibrium, if he is the holder of the j-title. To receive such representation, we should return to the formula (2.61) and determine the expectation. After that, to receive the price of the securities, we substitute it into (2.62.) This will give us, after the elementary transformation the following:

$$E(\overline{\alpha}_j) = \alpha_f + \tau(\dot{\alpha}_j, \dot{\alpha}_m). \qquad (2.64)$$

The obtained representation we shall call as the line of the securities market.

Profitability that can be expected by the investor, who is engaged in risky investments, will be formed from the risk-free interest rate and the risk premium. The latter, in its turn, is the product of the risk market price and the risk values of the given object of investments. The risk value is measured by the scalar product of the profitability rate of investments and the profitability rate of change of the market portfolio $(\dot{\alpha}_j, \dot{\alpha}_m)$.

Substitution of (2.63) into (2.64) will directly lead to:

$$E(\overline{\alpha}_j) = \alpha_f + \underbrace{(E(\overline{\alpha}_j) - \alpha_f)}_{\text{superprofitability}} \cdot \tau_j. \qquad (2.65)$$

In other words, to tau in VET. It is clear that tau shows the reaction of profitability of the j-financial title to the change of profitability in the market of financial titles $E(\overline{\alpha}_m)$ as the whole. Tau that may be equal to, for example, 0.9, means that, if the expected profitability of the market portfolio is increased by 1 %, then the expected profitability of the j-title is increased only by 0.9 %. In (2.65), tau is multiplied not only by the expected profitability of the market portfolio, but also by the negative interest rate.

Tau has one property that is extremely useful at shaping the portfolio that includes risky financial titles. If we designate the tau of one portfolio as symbol τ_p and the tau of the risky financial title as symbol τ_j, then we'll have the following:

$$\tau_p = \sum_{j=1}^{J} \omega_j \tau_j, \tag{2.66}$$

and ω_j here designates the share of the j-title in the portfolio p. The tau of the portfolio is always the linear combination of tau values of the securities, and they should be evaluated with relative shares of the portfolio. The regularity (2.66) can be easily shown. For this purpose, we shall start with the definition:

$$\tau_p = \frac{(\dot{\alpha}_p, \dot{\alpha}_m)}{(\dot{\overline{\alpha}}_m)^2}.$$

It is obvious that the aleatory variable $\overline{\alpha}_p$ is the linear combination of profitabilities that comprise the portfolio:

$$\overline{\alpha}_p = \sum_{j=1}^{J} \omega_j \overline{\alpha}_j,$$

therefore, the following is true:

$$\tau_p = \frac{\sum_{j=1}^{J} \omega_j (\dot{\alpha}_j, \dot{\alpha}_m)}{(\dot{\overline{\alpha}}_m)^2} = \frac{\omega_1 (\dot{\alpha}_1, \dot{\alpha}_m)}{(\dot{\overline{\alpha}}_m)^2} + \ldots + \frac{\omega_J (\dot{\alpha}_J, \dot{\alpha}_m)}{(\dot{\overline{\alpha}}_m)^2} = \omega_1 \tau_1 + \ldots + \omega_J \tau_J,$$

and that corresponds to (2.66).

2.13. Analysis of equilibrium

Now, we shall find out, whether there is any difference between the specialized investor and all other investors. First of all, we are interested in the total number of risky financial titles in the portfolios of managing subjects and their distribution among these subjects. To answer this question, we shall presume that there is equilibrium. Equilibrium in the market of securities can be describe through:

1) system of prices $p(r_0), p(\overline{r}_1), \ldots, p(\overline{r}_j)$ and

2) number of financial titles $n_0^i, n_1^i, \ldots, n_J^i$, which are of demand with the investors, who try to implement their optimal plans.

If we want equilibrium, the demand should exactly meet the supply. Therefore,

$$\sum_{i=1}^{I} n_j^i = \sum_{i=1}^{I} \bar{n}_j^i \text{ for each } j. \quad (2.67)$$

It is easy to prove that according to the Valras's law, equilibrium in the market of financial titles always means equilibrium in the market of consumer benefits. For this purpose, we shall return to the formula (2.41)

$$C_0^i + \frac{n_0^i}{1+\alpha_f} + \sum_{j=1}^{J} n_j^i\, p(\bar{r}_j) = \bar{C}_0^i + \frac{\bar{n}_0^i}{1+\alpha_f} + \sum_{j=1}^{J} \bar{n}_j^i\, p(\bar{r}_j)$$

and sum for all investors. If the price of the secured money flow equal to $r_0 = 1$ is designated by the symbol $p(r_0)=1/(1+\alpha_f)$ and the additives are permuted, then we shall have:

$$\sum_{i=1}^{I} C_0^i - \sum_{i=1}^{I} \bar{C}_0^i = \sum_{i=1}^{I}\sum_{j=0}^{J} \bar{n}_j^i\, p(\bar{r}_j) - \sum_{i=1}^{I}\sum_{j=0}^{J} n_j^i\, p(\bar{r}_j).$$

Equilibrium in the markets of financial titles means that the right member of this formula becomes equal to zero. However, this would at once means that the supply of consumer benefits at the point of time t=0 should meet the demand. Otherwise, the left-hand member of the formula would not accept the zero value. With reference to the market of consumer benefits, at the point of time t = 1, it is possible, by referring to (5.11) to find appropriate argumentation.

Now, under condition that all markets are in equilibrium, it is possible to prove two important theorems.

2.14. Diversification

The first theorem states it true

$$n_j^i > 0 \quad (2.68)$$

for all i-investors and for all j-securities.

Each individual's demand for risky financial titles in equilibrium is positive. And, to be more exact: each investor holds risky financial titles of each existing types. There are no risky securities, which are rejected by any investor. For this reason, we may say that each of them is realizing the strategy of mixed risk or diversification.

Proof. To prove, it is necessary to recollect the formula (2.50). With reference to the i-investor, it has the following form:

$$E(\bar{r}_j)-(1+\alpha_f)p(\bar{r}_j)=-2\frac{\partial L^i/\partial \dot{r}^2(\overline{C}_1)}{\partial L^i/\partial E(\overline{C}_1)}\cdot\sum_{k=1}^{J}n_k^i(\dot{r}_j,\dot{r}_k),$$

and when using:

$$h^i=-2\frac{\partial L^i/\partial \dot{r}^2(\overline{C}_1)}{\partial L^i/\partial E(\overline{C}_1)} \qquad (2.69)$$

it is possible to reduce to:

$$E(\bar{r}_j)-(1+\alpha_f)p(\bar{r}_j)=\sum_{k=1}^{J}h^i n_k^i(\dot{r}_j,\dot{r}_k).$$

Therefore, for J financial titles, it is possible to receive the following at matrix notation

$$\begin{pmatrix} E(\bar{r}_1)-(1+\alpha_f)p(\bar{r}_1) \\ E(\bar{r}_2)-(1+\alpha_f)p(\bar{r}_2) \\ \cdots\cdots \\ E(\bar{r}_j)-(1+\alpha_f)p(\bar{r}_j) \end{pmatrix}=\begin{pmatrix} (\dot{r}_1,\dot{r}_1)\ldots(\dot{r}_1,\dot{r}_J) \\ (\dot{r}_2,\dot{r}_1)\ldots(\dot{r}_2,\dot{r}_J) \\ \cdots\cdots \\ (\dot{r}_J,\dot{r}_1)\ldots(\dot{r}_J,\dot{r}_J) \end{pmatrix}\cdot\begin{pmatrix} h^i n_1^i \\ h^i n_2^i \\ \cdots \\ h^i n_J^i \end{pmatrix}.$$

If we recognize that the matrix of scalar products it is nondegenerate, then the following is true:

$$\begin{pmatrix} (\dot{r}_1,\dot{r}_1)\ldots(\dot{r}_1,\dot{r}_J) \\ (\dot{r}_2,\dot{r}_1)\ldots(\dot{r}_2,\dot{r}_J) \\ \cdots\cdots \\ (\dot{r}_J,\dot{r}_1)\ldots(\dot{r}_J,\dot{r}_J) \end{pmatrix}^{-1}\cdot\begin{pmatrix} E(\bar{r}_1)-(1+\alpha_f)p(\bar{r}_1) \\ E(\bar{r}_2)-(1+\alpha_f)p(\bar{r}_2) \\ \cdots\cdots \\ E(\bar{r}_j)-(1+\alpha_f)p(\bar{r}_j) \end{pmatrix}=\cdot\begin{pmatrix} h^i n_1^i \\ h^i n_2^i \\ \cdots \\ h^i n_J^i \end{pmatrix}.$$

From here follows the simple argumentation. As it was presumed that expectations of all participants of the market are similar, the left member of the above system of equations for everyone should be similar, as it has only scalar products of the recurrent flows of risky titles, expected money flows and cost of these securities, and also the risk-free interest rate. Managing subjects of all the above values, under supposition, have the same opinion. But, if the left members of the system of equations are independent of investors, then and the right members should be the same for everyone.

Therefore, for any two participants of the market i_1 and i_2, the following should be true:

$$h^{i1} n_j^{i1} = h^{i2} n_j^{i2} \ \forall \ j. \qquad (2.70)$$

Let's recollect that we have started with the premises of disinclination to risk and dissatisfaction of needs of each of the investors. This means that

$$\frac{\partial L^i}{\partial E(\overline{C}_1)} > 0 \quad \text{for each i}$$

and

$$\frac{\partial L^i}{\partial \dot{r}^2(\overline{C}_1)} < 0 \quad \text{for each i,}$$

then

$$h^i = -2\frac{\partial L^i / \partial \dot{r}^2(\overline{C}_1)}{\partial L^i / \partial E(\overline{C}_1)} \quad \text{for each i.}$$

For the purpose of equilibrium in the market of securities, it is required that the offered number of financial titles of the determined type $\sum_{i=1}^{I} \overline{n}_j^i$ was the same as the demand. Therefore, at least one the investor should have positive demand for the title j. But, due to (2.70) and $h^i > 0$ from the positiveness of demand for the unique investor, it is true for all investors that $n_j^i > 0$. Thus, the theorem is proved.

We have just proved that each investor in optimum holds a positive number of each of the existing risky financial titles. This does not mean that investors present their demand for risky securities in a rather individual way. You see, all of them have no identical utility functions, and therefore, it might happen that one the investors may have the greater demand to a certain financial title, than to another. But, it is impossible under our suppositions. If we designate the relative share of risky portfolio in possession of the i-participant of the market, by the symbol

$$\omega_j^i = \frac{n_j^i p(\bar{r}_j)}{\sum_{k=1}^{J} n_k^i p(\bar{r}_k)}, \tag{2.71}$$

then the theorem that will be proved now states that these shares are the same for all investors, ω_j^i is independent of i. This can be written down as follows:

$$\omega_j^i = \omega_j. \tag{2.72}$$

It is true for all i-investors and all j-securities.

Thus, ω_j is the share of "participation" of the j-securities in the risky part of the portfolio of every investor. Such uniformity of structures of the risky part of the portfolio is very surprising. There is an impression that all risky financial titles were in the mutual fund of securities (with the share of ω_j), from which separate investors buy a certain part. But, this part of the fund of securities is also the totality of the risky part of the portfolio of investors. They, in addition, do not purchase any risky titles. Therefore, it is possible to say that (5.43) describes the mutual fund theorem.

Proof. For the realization of the proof, we, first of all, should determine the inverse matrix of the scalar product:

$$\begin{pmatrix} \partial_{11} \ldots \partial_{1J} \\ \partial_{21} \ldots \partial_{2J} \\ \ldots\ldots\ldots \\ \partial_{J1} \ldots \partial_{JJ} \end{pmatrix} = \begin{pmatrix} (\dot{r}_1, \dot{r}_1) \ldots (\dot{r}_1, \dot{r}_J) \\ (\dot{r}_2, \dot{r}_1) \ldots (\dot{r}_2, \dot{r}_J) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (\dot{r}_J, \dot{r}_1) \ldots (\dot{r}_J, \dot{r}_J) \end{pmatrix}^{-1}.$$

Thus, we may lay down the set of equations (5.40) in the form

$$\underbrace{\sum_{k=1}^{J} \vartheta_{jk} \left(E(\bar{r}_k) - (1+\alpha_f) E(\bar{r}_k)\right)}_{:=\Theta_j} = h^i n_j^i \quad \text{for each } j. \tag{2.73}$$

Pay attention to the fact that the left member of these equations is the same for all investors. This follows from the premise of the similarity of expectations. If basing on this, to express the demand for the i-type securities of the j-participant of the market, we have:

$$n_j^i = \frac{\Theta_j}{h^i} \text{ for each } j. \tag{2.74}$$

The number of j-securities in demand of the i-participant of the market is calculated, therefore, in such a way that the identical to all investors value Θ_j is divided by the number h^i that reflects the specific relation of this investor to risk. If we substitute the result into the differential equation (2.71), then we shall received:

$$\omega_j^i = \frac{\frac{\Theta_j}{h^i} p(\bar{r}_j)}{\sum_{k=1}^{J} \frac{\Theta_k}{h^i} p(\bar{r}_k)}.$$

Obviously, the formula can be cancelled to the number h_i that designates a certain investor, and after that remains the following:

$$\omega_j^i = \frac{\Theta_j p(\bar{r}_j)}{\sum_{k=1}^{J} \Theta_k p(\bar{r}_k)}. \tag{2.75}$$

The proof is completed, as the right member of the equation is also the same for all investors.

The formula (2.75) enables us to calculate the coefficient of the mutual fund structure. It allows calculating, what relative share of his risky invested property is intended for the financial title j by each decision making person. Under condition that the investor is capable of calculating the profitability rate of change and the scalar product of PRC, and also the expected profitability and the risk-free interest rate, we may deduce, from (2.75), the formula of ω_j computation that wil be convenient for application.

As, between the money flow of risky financial titles, their prices and profitability, by definition, there exists the relation:

$$\bar{\alpha}_j = \frac{\bar{r}_j}{p(\bar{r}_j)} - 1,$$

then the scalar product of profitability of two financial titles can be presented as follows:

$$\left(\dot{\alpha}_j, \dot{\alpha}_k\right) = \frac{\left(\dot{r}_j, \dot{r}_k\right)}{p\left(\dot{r}_j\right) \cdot p\left(\dot{r}_k\right)}.$$

If we express the scalar products of profitability in the form of the matrix and then transform this matrix into the inverse one, we shall receive:

$$\begin{pmatrix} \theta_{11} \ldots \theta_{1J} \\ \theta_{21} \ldots \theta_{2J} \\ \ldots\ldots\ldots\ldots \\ \theta_{J1} \ldots \theta_{JJ} \end{pmatrix} = \begin{pmatrix} (\dot{\alpha}_1, \dot{\alpha}_1) \ldots (\dot{\alpha}_1, \dot{\alpha}_J) \\ (\dot{\alpha}_2, \dot{\alpha}_1) \ldots (\dot{\alpha}_2, \dot{\alpha}_J) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\dot{\alpha}_J, \dot{\alpha}_1) \ldots (\dot{\alpha}_J, \dot{\alpha}_J) \end{pmatrix} = \begin{pmatrix} \partial_{11} p(\bar{r}_1) p(\bar{r}_1) \ldots \partial_{1J} p(\bar{r}_1) p(\bar{r}_J) \\ \partial_{21} p(\bar{r}_2) p(\bar{r}_1) \ldots \partial_{2J} p(\bar{r}_2) p(\bar{r}_J) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \partial_{J1} p(\bar{r}_J) p(\bar{r}_1) \ldots \partial_{JJ} p(\bar{r}_J) p(\bar{r}_J) \end{pmatrix}.$$

Numerator of the right member of the formula (2.75) can be laid down by using (2.73),

$$\Theta_j p(\bar{r}_j) = p(\bar{r}_j) \sum_{k=1}^{J} \vartheta_{jk} \left(Ep(\bar{r}_k) - (1 + \alpha_f) p(\bar{r}_k)\right) = p(\bar{r}_j) \sum_{k=1}^{J} \vartheta_{jk} p(\bar{r}_k) \times$$

$$\times \left(E(\bar{\alpha}_k) - \alpha_f\right) = \sum_{k=1}^{J} \underbrace{\vartheta_{jk} p(\bar{r}_k)}_{:=\theta_{jk}} \cdot \left(E(\bar{\alpha}_k) - \alpha_f\right) = \sum_{k=1}^{J} \theta_{jk} \cdot \left(E(\bar{\alpha}_k) - \alpha_f\right).$$

Substitution into (2.75) will result (at already justified refusal from the notation that designate a certain investor) in the following:

$$\omega_j = \frac{\sum_{k=1}^{J} \theta_{jk} \cdot \left(E(\bar{\alpha}_k) - \alpha_f\right)}{\sum_{l=1}^{J} \sum_{k=1}^{J} \theta_{lk} \cdot \left(E(\bar{\alpha}_k) - \alpha_f\right)}.$$

<u>The updated Tobin's theorem of separation</u>. The instructive result, at which all individuals hold their risky portfolio of securities with similar structure (in the state of market equilibrium) independent of their degree of aversion of risk, is named as the portfolio separation or the Tobin's separation. The portfolio structure consisting of risky financial titles may be determined independent of separate investors' behaviour.

To prevent misunderstanding, we should point out that this thesis is real only for one, although essential, part of decisions of the necessity of investments or savings. The investors in the model analyzed here invest their money partially into the risk-free assets (at interest rate α_f) and partially into the risky assets (at the expected market profitability $E(\overline{\alpha}_m)$). The risk-free part of investments depends (and very much strongly) on their attitude to risk. Thus, the following is true: the higher the degree of aversion of risk, the more means are invests at the risk-free interest rate.

Soundness of this thesis can be shown, if we use the condition of the first order that has not been considered till now (2.48)

$$\left(C_0^i + \frac{n_0^i}{1+\alpha_f} + \sum_{j=1}^{J} n_j^i p(\overline{r}_j) - \overline{C}_0^i - \frac{\overline{n}_0^i}{1+\alpha_f} - \sum_{j=1}^{J} \overline{n}_j^i p(\overline{r}_j) \right) = 0$$

with reference to a certain decision-making person. If this formula is expressed through the size of the investor's savings at the point of time t = 0 and designate it as S^i, we shall receive:

$$S^i = \frac{n_0^i}{1+\alpha_f} + \sum_{j=1}^{J} n_j^i p(\overline{r}_j) = \overline{C}_0^i + \frac{\overline{n}_0^i}{1+\alpha_f} + \sum_{j=1}^{J} \overline{n}_j^i p(\overline{r}_j) - C_0^i .$$

(2.76)

If to treat the decision on the size of the today's consumption as the set one, then, thus, the total size of savings S^i is simultaneously fixed. We still have one problem to resolve: what part of investments will be invested into the risk-free and risky assets. To make it clear, we shall substitute (2.69) and (2.74) into (2.76). Then, after appropriate transformations, total investments will be equal to:

$$S^i = \underbrace{\frac{n_0^i}{1+\alpha_f}}_{\text{risk-free investment}} + \underbrace{\left(-\frac{1}{2} \cdot \frac{\partial L^i / \partial E(\overline{C}_1)}{\partial L^i / \partial \dot{r}^2(\overline{C}_1)} \cdot \Theta_j p(\overline{r}_j) \right)}_{\text{risky investment}}$$

It is obvious that the part of means that will be invested into risky assets depends on the individual rate of substitution:

$$\frac{\partial L^i / \partial E(\overline{C}_1)}{\partial L^i / \partial \dot{r}^2(\overline{C}_1)} .$$

With the help of (2.17) and (2.16) we should understand that this expression for each investor, whose behaviour is characterized by aversion of risk and unsatisfied needs, should be negative and the less by its sum the more his aversion of risk. It means that the size of their risky investments is the smaller, the more his disinclination to risk. Accordingly, there will be more investments into risk-free financial titles, if total savings S^i are treated as the given value.

2.15. Theory of the corporation capital structure

In countries with market economy, the corporate form of a company business has found its application. It differs by existence of two sources of the company business financing: issue of shares (raising of means of holders) and borrowing (raising of loan proceeds).

Loan proceeds may be raised by the bank credit or by issue of debenture stocks in the form of bonds.

The cost of the company's capital consists of two items:

$$V = S + B,$$

where, S - cost of the ownership capital,

B - cost of the debt.

The relation of the cost of the debt to the cost of the fixed capital (B/S) has received the name of coefficient of financial leverage in scientific and economic literature.

Each of forms of the company's capital financing is connected within certain kinds of disbursements. They are dividends paid to the proprietors of the company as compensation for their invested funds, and the interest for credits or bonds to creditors of the company.

Costs of creation of the company's capital is reflected by the index of the weighted average costs of the capital r_c:

$$r_c = \frac{S}{V} r_S + \frac{B}{V} r_B,$$

where, S/V - share of the ownership capital,

B/V - share of the borrowed capital in the structure of the company's capital.

Borrowing of funds is objectively less risky in comparison with investments into ownership capital. First, because debt interests, as the rule, are specified beforehand by the terms of the credit agreement or by the prospectus of emission of bonds, whereas the shares income is not known beforehand and depends on the results of the corporation activity. It is natural, that in both case there is a risk of that the corporation, for this or that reason, will not provide payments of interests on debt or the part of the income on the fixed capital. But, and in the latter case, borrowing is less risky owing to existence of the right of priority of creditors on the company's assets guaranteed by the economic legislation of the absolute majority of countries. According to this principle, in case the company terminates its activity, the creditors' requests on return of the debt will be satisfy the first. Proprietors - shareholders have residual requests. The part of assets, for which they have the right to claim, is equal to the residual cost after payments according to the requests of all creditors.

According to this approach, there is some optimum level of the financial leverage that guarantees minimum costs of capital financing, and each company has it own optimal structure of capital it tries to achieve for the purpose of lower costs.

However, the traditional theory has some very essential drawbacks: it does not take into account existence of markets of capital at all and is not the model of equilibrium.

M.Miller's and F.Modiljani's model is an important step in development of the modern theory, according to which:

1) Structure of the capital does not affect the size of weighted-average costs of capital;

2) Size of dividends does not affect the cost of the corporation shares.

At the same time, the traditional theory of corporate finances and the theory of Miller and F.Modiljani very often contradicts to empirical data referring to objective laws of choosing the source of financing, dividends policy and other kinds of activity of companies.

One from the most important conclusions of many empirical researches: all over the world, the main source of financing of the company's investments is the retained income (instead of the external financing in the form of debt or issue of shares). Practically everywhere, the source of external financing is bank credits (excluding, perhaps, such countries, as the USA and Great Britain), whereas the market of capital plays a rather small role in financing of investment projects.

The size of the company influences the form of fund raising: the bigger the company, the easier for it to issue securities and to raise funds by this way. Only the largest corporations are raising funds by issue of bonds. In Europe, large companies place loans through emission of eurobonds, instead of in the national markets, as this mode is much cheaper. However, in comparison with bank financing of large projects, which is provided by bank syndicates, emission of euro-bonds is insignificant.

For small enterprises, bank credits, practically, are the only source of financing (we speak about external financing). Penetration into the market of securities is rather expensive for small and average companies. Only losses at the initial allocation of because of their undervaluation may reach 20 % from the real cost of shares!

Let's consider the medium-sized companies. They cannot be characterized unambiguously; therefore, we shall compare the North America and Great Britain with the continental Europe and Japan. There is a large difference between these countries (referring to the behaviour of the medium companies).

For example, medium companies in the countries of the North America and Great Britain tend to penetrate (or to get out) into the stock market with the purpose of placement of bonds and shares among private investors. Besides, there are certain expenditures on entering into the listing of the exchange.

The same tendency of the medium companies in countries of Europe and Japan, on the contrary, are practically not traceable.

All the above-stated prove the necessity of development of new theoretical approaches to the description of the optimal structure of the company's capital.

1. We analyze the economy at two points of time t = 0 ("today") and t = 1 ("in one year"). The argumentation in the model with two point of time is very simple. But, we may easily loosen this strict assumption.

2. Referring to the point of time t = 1, there exists an uncertainty. It is manifested, first of all, by the fact the income from investments that can be carried out by the company cannot be closely predicted. At the point of time t = 1, there are may take place different situation s = 1..., S, and it is presumed that their number is finite.

3. There is the large number of investors, who, at the point of time t = 0, propose their savings to companies by purchasing financial titles. Such titles may be in the form of securities of proprietors (shares) or creditors (bonds). The company uses the income from sale of these securities for financing of its investments. In its turn, it grants to those,

who offer the capital, payments due at the point of time t = 1. These payments are covered by the income from investments, and at the same time, the creditors are promised the fixed income, whereas they promise to the shareholders that they will receive the sum that will remain from investment income after payments to the creditors. If the size of investment income is rather small, creditors will receive the whole investment income, and shareholders will receive nothing. Nevertheless, the amount that may be distributed between all investors does not exceed the proceeds from investments at the point of time t = 1. Therefore, the civil responsibility is excluded.

4. All financial titles are circulating in the uninterruptedly functioning market of capital. It means, there are no neither transaction costs connected with sale and purchase of securities, nor taxes, nor restriction of trade or access to the market. This means that sales without cover are possible.

5. Market of capital is competitive. Nobody has the leading position. Therefore, for everyone, the prices of securities are given.

6. All participant of the market have similar views on the amount of proceeds, which can be expected by them possession of financial titles, if at the point of time t = 1 s-situation occurs. Therefore, uniform expectations of the future are expected. But, this uniformity does not necessarily refers to probabilities of situations at the point of time t = 1.

7. Possibilities of arbitration do not exist.

8. There are two companies, which differ only by the structure of their capital. These companies are designated by indexes L and U, and L designates the company that is partially financed by the borrowed capital (in English: levered company), U - company without such financing I (in English: unlevered company). This assumption is very important. It means that the structure of capital is the only factor that differs these two companies. Therefore, they pursue the same investment policy, issue the same products at the same costs and sell their products at the same condition and in the same markets. Such coincidence is possible only in experiments. But we need it, as otherwise we would not be able to evaluate, whether the cost of the company depends on the structure of its capital or not.

Let's introduce the following designations:

V_0 - market value of the whole company;

E_0 - market valuation (cost) of the ownership capital (value of equity);

$\bar{E}_1$ - unsecured requests of the proprietors at the point of time t = 1;

D_0 - market valuation (cost) the borrowed capital (value of debt);

$\bar{D}_1$ - unsecured requests of the creditors at the point of time t = 1;

$\bar{X}$ - unsecured investment income at the point of time t = 1.

As it is accepted, we shall designate all unsecured values (i.e. values that are formed under uncertainty) by the tilde. As the company's activity U is not financed by the borrowed capital, $D_0^U = 0$ and $V_0^U = E_0^U$ is true. On the contrary, for the company L , the identity $V_0^L = E_0^L + D_0^L$ is observed. As both companies, except for the structure of their capital, are identical, then the following is true:

$$\bar{r}^U = \bar{r}^L \tag{2.77}$$

for both companies.

2.15.1. VET and optimal structure of the company's capital

The simple method of researches of the optimal structure of capital is based on the use of the price representation of VET. For this purpose, we shall recollect this representation of the model (2.60) with the following form:

$$p(\bar{r}) = \frac{E(\bar{r}) - \tau(\dot{r}, \dot{\alpha}_m)}{1 + \alpha_f}.$$

For computation of the equilibrium price of one financial title that at the point of time t = 1 "promises" secured payments, it is necessary to determine, first of all, the expected value of these payments (future proceeds), then, to correct calculation with reference to the risk by decreasing of the expected value by the product of the risk market price and the scalar product between payments for the financial title and market profitability. Now we use this formula for market cost valuations of the ownership and borrowed capital.

To do this, it is necessary to toughen the above assumptions. We presumed that all participant of the market have uniform expectations of the future, but did not presume that probabilities of situations at the point of time t = 1 are equally evaluated. Now, we need to presume it, otherwise, conditions of VET will not be fulfilled.

Further toughening of the above evaluated happen due to the use of VET.

First, we may state that the market value of the company U that does not resorting to the borrowed capital, may be determined by the formula:

$$V_0^U = E_0^U = \frac{E(\bar{r}) - \tau(\dot{r}, \dot{\alpha}_m)}{1 + \alpha_f}. \qquad (2.78)$$

Risk-free indebtedness to the creditors. Let's presume that the promise to pay creditors is given only to the amount that in the point of time t = 1 can be securely paid. At risk-free indebtedness, the market value is equal to:

$$D_0^L(t) = \frac{D_1(t)}{1 + \alpha_f}. \qquad (2.79)$$

In this formula, the empirical fact has found its reflection. The market value of bonds is determined not only by its interest rate, but also by its demand and supply according to the formula:

$$D_0^L = \frac{D_1}{1 + \alpha_f}, \qquad (2.80)$$

where D_0^L – expectation D_0^L (t).

As it is presumed that creditors' requests are satisfied exclusively at the expense of investment income and the civil liability is excluded, the following is true:

$$\begin{gathered} \bar{E}_1 + D_1 = \bar{r}, \\ E(\bar{E}_1) + E(D_1) = E(\bar{r}), \\ \dot{\bar{E}}_1 + \dot{D}_1 = \dot{\bar{r}}. \end{gathered} \qquad (2.81)$$

Substitutions and transformation allow finding the difference between the market value of capital of the company that does not resort to the borrowed capital, and the market value of the company with the borrowed capital:

$$V_0^U - V_0^L = E_0^U - E_0^L - D_0^L,$$

$$V_0^U - V_0^L = \frac{E(\bar{r}) - \tau(r, \alpha_m)}{1+\alpha_f} - \frac{E(\bar{E}_1) - \tau(\dot{E}_1, \dot{\alpha}_m)}{1+\alpha_f} - \frac{E(D_1)}{1+\alpha_f},$$

$$(\dot{r}, \dot{\alpha}_m) = (\dot{\bar{E}}_1, \dot{\alpha}_m) + (\dot{\bar{D}}_1, \dot{\alpha}_m), \quad \bar{E}_1 + D_1 = \bar{r},$$

$$V_0^U - V_0^L = \frac{E(\bar{r}) - E(\bar{E}_1) - E(D_1)}{1+\alpha_f} - \frac{\tau\left[\bar{E}_1\dot{\alpha}_m + \dot{\bar{D}}_1\dot{\alpha}_m - \bar{E}_1\dot{\alpha}_m\right]}{1+\alpha_f}, \quad (2.82)$$

$$\mathbf{V_0^U - V_0^L = -\frac{\tau(\dot{\bar{D}}_1\dot{\alpha}_m)}{1+\alpha_f}; \quad V_0^L - V_0^U = \frac{\tau(\dot{\bar{D}}_1\dot{\alpha}_m)}{1+\alpha_f}.}$$

Here, three cases are possible:

1) $(\dot{\bar{D}}_1, \dot{\alpha}_m) > 0, \quad V_0^L > V_0^U;$

2) $(\dot{\bar{D}}_1, \dot{\alpha}_m) = 0, \quad V_0^L = V_0^U;$ (Miller and Modilyunis case);

3) $(\dot{\bar{D}}_1, \dot{\alpha}_m) < 0, \quad V_0^L < V_0^U,$

where V_0^L – value of the company with the borrowed capital;
V_0^U – value of company without the borrowed capital.

SUMMARY TO SECTION II

1. The value that measures the degree of risk of funds investments into securities is the square of profitability rate of change (PRC):

$$\left(\frac{dr}{dt}\right)^2,$$

where $\frac{dr}{dt} = \frac{\Delta r}{\Delta t} = \frac{\text{Change of profitability}}{\text{Period of time}}$ at $\Delta t \to 0$.

2. Application of the square of PRC as the unit of risk measurement instead of the standard deviation allows to get rid of the necessity to use the quadratic form of the utility function:

$$u(\omega) = \alpha\omega - \beta\omega^2, \quad \alpha, \beta > 0.$$

3. Use of the square of PRC as the unit of risk measurement allows to consider securities, which have:

1) Deviation of profitability that differs from normal;

2) Value of variance that changes in time.

4. Instead of the principle of Neumann - Morgenstern (according to which the individual under risk make his choice basing on the maximization of the expected utility of the result), the person chooses such distribution P from the set of alternatives that:

$$V(p) = \max_P V(p) = \max_P E(p)[u(\omega)].$$

The principle according to which an individual distributes his available financial resources at the point within time interval $[t_1, t_2]$ in such a way so that the functional $S = \int_{t_1}^{t_2} (R - U) dt$ at $t_1 < t < t_2$ was minimum, where R - risk of investment, U - utility function.

5. Principle of the investor's optimal behaviour can be represented by the equality to the zero variation:

$$\delta \int_{t_1}^{t_2} L\left(r, \frac{dr}{dt}, t\right) dt = 0.$$

6. It was offered to use as the index that determines the degree of correlation between profitability of various investment decisions, the

cosine of the averaged angle between the vectors of rates of change of profitability:

$$\cos\left(\frac{d\bar{r}_1}{dt} \wedge \frac{d\bar{r}_2}{dt}\right) = \frac{\left(\frac{d\bar{r}_1}{dt}, \frac{d\bar{r}_2}{dt}\right)}{\left\|\frac{d\bar{r}_1}{dt}\right\| \cdot \left\|\frac{d\bar{r}_2}{dt}\right\|}.$$

At $\cos\left(\frac{d\bar{r}_1}{dt} \wedge \frac{d\bar{r}_2}{dt}\right) = 1$ assets are characterized by completely positive correlation; any increase (decrease) in profitability of one of them will lead to increase (decrease) in profitability of the other.

At $\cos\left(\frac{d\bar{r}_1}{dt} \wedge \frac{d\bar{r}_2}{dt}\right) = -1$ assets are characterized by completely negative correlation; any increase (decrease) is profitability of one of them will lead to decrease (increase) in profitability of the other.

At $\cos\left(\frac{d\bar{r}_1}{dt} \wedge \frac{d\bar{r}_2}{dt}\right) = 0$ profitability of one asset it is not connected with profitability of the second.

7. Markovitz's model, under condition that we use the square of PRC for measurement of risk, is represented as follows:

$$\min_{x_1, x_2, \ldots, x_n} \left\{\left(\frac{dr_p}{dt}\right)^2\right\} = \min_{x_1, x_2, \ldots, x_n} \left\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{dr_i}{dt} \cdot \frac{dr_j}{dt}\right\}$$

at limitations

$$\sum_{i=1}^{n} x_i = 1, \quad \sum_{i=1}^{n} \bar{r}_p x_i = \bar{r}_p.$$

8. Price of the j-financial title with the unsecured income, if the investor tries to make the compromise choice "consumption-investments" is determined by the expression:

SECTION III. BASIC POSTULATES OF THE WAVE VARIATIONAL ECONOMIC THEORY (WVET)

3.1. Mathematical apparatus of the wave variational economic theory

The base of the system of WVET is the principle of superposition:

If any economic system is capable is being in the condition tt is represented by t wave function ψ_1, and in another condition ψ_2, it may be in the condition that is represented by the wave function ψ, namely:

$$\psi_1 = c_1\psi_1 + c_2\psi_2,$$

where c_1 and c_2 - arbitrary, generally speaking, complex numbers which define amplitudes and phases of separate conditions ψ_1 and ψ_2.

From here follows that, if there is a series of possible conditions of the system, which are distinguished from each other only by some value (profitability, profitability rate of change and etc.) and are represented by wave functions, ψ_1, ψ_2, …, ψ_n, according to the principle of superposition, there exists some compound conditions:

$$\psi_1 = c_1\psi_1 + c_2\psi_2 + \ldots + c_n\psi_n, \tag{3.1}$$

where c_1, c_2, …, c_n - arbitrary, complex amplitudes.

Therefore, conditions should be described by such mathematical values, which can be added among themselves, multiplied by complex numbers and to receive, at the same time, values of the same series. It means that conditions should be compared with vectors of a certain linear space. Their role may be fulfilled by the complex quadratically integrable functions (ψ-functions) that are the vectors of the Hilbert space.

3.1.1. Symbols of Dirac

Any vector ψ of the Euclidean n-dimensional space is univalently set by the set of components in the fixed basic set. It is convenient to set these components in columns, noting the vector ψ in the form of the column matrix:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}.$$

Let's introduce the formally conjugate space of vectors ψ^+, which are received from ψ by hermitian conjugate, i.e. represent the row matrixes:

$$\psi^+ = (\psi_1^*, \psi_2^*, ..., \psi_n^*) \tag{3.2}$$

From this definition follows,

$$(\alpha\psi)^+ = \alpha^*\psi^+, \tag{3.3}$$

where α - any complex number.

Applying the rule that sets the product of the row matrix and the column matrix, hermitianly conjugated to it, we, in abbreviated form, may write down the formula (in matrix symbols) for the scalar product of vectors ψ and φ:

$$(\varphi, \psi) = \varphi^+\psi. \tag{3.4}$$

The relation (3.3) turns to the property of antilinearity of the scalar product according to the first argument.

Dirac has offered an extremely convenient system of symbols that sometimes even simplifies the proof of the various kinds of mathematical statements. We shall use it from now and resort to usual symbols only for the purposes of comparison.

So, according to Dirac, we shall designate

$$\psi \equiv |\psi\rangle, \quad \psi^+ \equiv \langle\psi| \tag{3.5}$$

and we shall name $|\psi\rangle$ as the ket vector, and $\langle\psi|$ - as the bra vector According to (3.1) and (3.2), formally these two types of vectors are connected by the operation of hermitian conjugate:

$$\langle\psi| \equiv |\psi\rangle^+, \quad |\psi\rangle \equiv \langle\psi|^+. \tag{3.6}$$

It should be underlined that ket vectors and bra vectors belong to different spaces, so they cannot by added.

However, with the help of the introduced symbols, it is still possible to construct more complicated formations, among which are the following elementary five types:

$$|\psi\rangle; \quad \langle\psi|; \quad \langle\varphi|\psi\rangle; \quad |\varphi\rangle\langle\psi|; \quad |\varphi\rangle|\psi\rangle.$$

The first, as it has already been said, is the ket vector or the vector of n-dimensional Euclidean space, the second “ is the bra vector or the vector of the conjugate space.

Let’s clarify the meaning of the third object. According to symbols (3.6), the symbol $\langle\varphi|\psi\rangle$ (the second dash inside is not written for short) represents the product of $\varphi^+\psi$. Taking into account (3.4), we see that $\langle\varphi|\psi\rangle$ is a simply number - the scalar product of vectors φ and ψ:

$$\langle\varphi|\psi\rangle \equiv (\varphi, \psi). \tag{3.7}$$

In particular, in these symbols, the norm of the vector ψ is noted as

$$\|\psi\| = \sqrt{\langle\psi|\psi\rangle}. \tag{3.8}$$

Let’s introduce in spaces of ket and bra vectors the matched orthonormalized bases, i.e. sets of such measuring vectors, which are received from each other by hermitian conjugate. These measuring unit vectors will be designated by us as $|1\rangle, |2\rangle, ..., |n\rangle$ and $\langle 1|, \langle 2|, ..., \langle n|$, accordingly. Then, the condition of the basis orthonormality can be written down as follows:

$$\langle j|k\rangle = \delta_{jk}. \tag{3.9}$$

Vectors $|\psi\rangle$ and $\langle\psi|$ can be expanded on their basis:

$$|\psi\rangle = \sum_{j=1}^{n} \psi_j |j\rangle; \quad \langle\psi| = \sum_{j=1}^{n} \overline{\psi}_j \langle j|. \tag{3.10}$$

can be expanded on their basis: $\langle k|$, and the second to the right – by $|k\rangle$, we shall receive

$$\psi_j = \langle j|\psi\rangle \tag{3.11}$$

and

$$\overline{\psi}_k = \langle \psi | k \rangle = \langle k | \psi \rangle^* = \psi_k^*,$$

i.e.

$$\overline{\psi}_j = \psi_j^*. \tag{3.12}$$

Sets of numbers ψ_j and $\overline{\psi}_j$ are the sets of vectors $|\psi\rangle$ and $\langle\psi|$, components, accordingly, and they are determined by similar methods.

Using of the received expressions for components of vectors $|\psi\rangle$ and $\langle\psi|$, we may expand them (3.10) and write down as follows:

$$|\psi\rangle = \sum_{j=1}^{n} |j\rangle\langle j|\psi\rangle \tag{3.13a}$$

and

$$\langle\psi| = \sum_{j=1}^{n} \langle\psi|j\rangle\langle j|. \tag{3.13b}$$

3.1.2. Linear operators in the Euclidean space

Operator is a certain rule F, with the help of which, from some mathematical object ψ, we can receive another object φ of the same nature.

Operator F is the rule, according to which each vector ψ of the Euclidean n-dimensional space is compared with the vector φ of the same space. Using the symbols of Dirac, we shall write down:

$$|\varphi\rangle = F|\psi\rangle \tag{3.14}$$

(symbol $\wedge$ used earlier above the symbol of the operator will be sometimes omitted by us).

Operator F is referred to as the linear one, if it satisfies the requirement:

$$F(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha(F|\psi\rangle) + \beta(F|\varphi\rangle) \tag{3.15}$$

for any pair of vectors $|\psi\rangle$ and $\langle\varphi|$ and for any complex numbers α and β.

The linearity requirement of operators in the WVET is dictated by the principle of superposition.

Examples.

1. We shall consider operator I, which compares any vector $|\psi\rangle$ with the same vector:

$$I|\psi\rangle = |\psi\rangle. \tag{3.16}$$

This operator is called unit or identity operator; it is linear one.

2. A null operator is an operator Θ that transfers each vector in the null vector:

$$\Theta|\psi\rangle = |0\rangle. \tag{3.17}$$

3. Let's presume that we have the arbitrary vector $|\psi\rangle$ with components $\psi_1, \psi_2, \ldots, \psi_n$ (column matrix ψ) and the square matrix A of the n×n order. We shall determine the vector $|\varphi\rangle$ with components $\varphi_1, \varphi_2, \ldots, \varphi_n$, which are set by equalities

$$\left.\begin{array}{l} \varphi_1 = a_{11}\psi_1 + a_{12}\psi_2 + \ldots + a_{1n}\psi_n, \\ \varphi_2 = a_{21}\psi_1 + a_{22}\psi_2 + \ldots + a_{2n}\psi_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_n = a_{n1}\psi_1 + a_{n2}\psi_2 + \ldots + a_{nn}\psi_n \end{array}\right\}$$

or in the short form

$$\varphi_j = \sum_{k=1}^{n} a_{jk}\psi_k. \tag{3.18}$$

In matrix symbols, it may be noted as:

$$\varphi = A\psi. \tag{3.19}$$

Thus, each matrix of the n×n order sets a certain operator in the n-dimensional Euclidean space. It is clearly that such operator is the linear one. In the following paragraph, we shall show that, at the same time, on the contrary, each linear operator in the n-dimensional Euclidean space corresponds to a certain quadratic matrix of the n×n order. This, actually, explains the importance of the role, which is played by matrixes in various mathematical constructions.

4. We shall return now to the construction $|\varphi\rangle\langle\psi|$, introduced in the previous item. Formally, by adding an arbitrary ket vector $|\chi\rangle$, to this symbol to the right, we shall receive $|\varphi\rangle\langle\psi|\chi\rangle$. Two latter multiplicands produce the number $\langle\psi|\chi\rangle$, and there appears the new ket vector that is proportional to $|\varphi\rangle$. Similarly, if to $|\varphi\rangle\langle\psi|$ we add to the left the arbitrary bra vector $\langle\chi|$, we shall receive the new bra vector that is proportional to $\langle\psi|$. Thus, $|\varphi\rangle\langle\psi|$ is the operator (linear), which, by operating to the right on the ket vector, transfers it into the new ket vector, and by operating to the left on the bra vector, transfers it into the new bra vector.

5. All the above allows to receive an extremely useful representation for the unit operator. Knowing its definition (3.3), we may identically write down the expansion (3.13) of the arbitrary ket vector $|\psi\rangle$ on the orthonormalized basis in the form of:

$$I|\psi\rangle = \sum_{j=1}^{n} |j\rangle\langle j|\psi\rangle.$$

By virtue of arbitrariness of the vector $|\psi\rangle$, we receive the following:

$$I = \sum_{j=1}^{n} |j\rangle\langle j|. \tag{3.20}$$

This relation is fundamental and can be frequently used in future. It is call the condition of the orthonormalized basis completeness.

As the simple example of completeness conditions application, we shall write it down with the help of the vector norm square $|\psi\rangle$:

$$\|\psi\|^2 = \langle\psi|\psi\rangle = \langle\psi|I|\psi\rangle = \sum_{j=1}^{n} \langle\psi|j\rangle\langle j|\psi\rangle = \sum_{j=1}^{n} \psi_j^* \psi_j = \sum_{j=1}^{n} |\psi_j|^2.$$

In the set of linear operators, one can carry out all possible algebraic operations.

The sum of operators F and G is an operator F + G that associates each vector $|\psi\rangle$ with the vector $F|\psi\rangle + G|\psi\rangle$:

$$(F+G)|\psi\rangle = (F|\psi\rangle) + (G|\psi\rangle). \tag{3.21}$$

Note. If $|\varphi\rangle$ and $\langle\psi|$ are treated as matrixes of (3.18) type, then $|\varphi\rangle\langle\psi|$ will be the tensorial product of these matrixes.

The product of the operator F by the number α is called operator αF that transfers the vector $|\psi\rangle$ in the vector $\alpha(F|\psi\rangle)$:

$$(\alpha F)|\psi\rangle = \alpha(F|\psi\rangle). \tag{3.22}$$

The product of operators F and G is operator FG, operation of which is equivalent to the sequential application to the vector $|\psi\rangle$ first by operator G, then by operator F:

$$(FG)|\psi\rangle = F(G|\psi\rangle). \tag{3.23}$$

It is important that the product of operators is generally noncommutative:

$$FG \neq GF. \tag{3.24}$$

Till now, we considered operators operating in the Euclidean space of ket vectors. Equally, it is possible to introduce operators that are operating in the conjugate space of bra vectors. We shall give you appropriate definitions, which play fundamental role in the WVET.

Let's presume that in space of ket vectors there is an operator F:

$$|\chi\rangle = F|\psi\rangle. \tag{3.25}$$

Let's introduce operator F^+ that operates in the conjugate space, transferring the bra vector $\langle\psi|$, corresponding to $|\psi\rangle$, and the bra vector $\langle\chi|$, corresponding to $|\chi\rangle$:

$$\langle\chi| = \langle\psi|F^+. \tag{3.26}$$

Operator F^+ shall be called by us as operator that is conjugated with F. By definition, the conjugate operator operates on bra vectors to the right.

By multiplying (3.25) at the left by $\langle\varphi|$, and (3.26) at the right by $|\varphi\rangle$ and by comparing the left members, which are complex conjugated due to the hermicity property of the scalar product, we shall receive main property of the conjugate operator

$$\langle\psi|F^{+}|\varphi\rangle^{*} = \langle\varphi|F|\psi\rangle, \tag{3.27}$$

which can be considered as its definition.

Let's find the relation using the ordinary symbols. For this purpose, we shall point out, first of all, that expression of the type $\langle\varphi|F|\psi\rangle$ we shall always understand as follows: vector ψ on the left is acting on the operator F, then, the received vector is multiplied scalarly from the left by φ. Thus, (3.27) in standard symbols can be written down as follows:

$$(\psi, F^{+}\varphi)^{*} = (\varphi, F\psi), \quad \text{or} \quad (F^{+}\varphi, \psi) = (\varphi, F\psi). \tag{3.28}$$

For example, if operator F operates in the space of complex functions $\psi(x_1, x_2, ..., x_N)$, where the scalar product is set by the formula:

$$(\varphi, \psi) = \int \varphi^{*}\psi\, dX$$

$dX \equiv dx_1 dx_2 ... dx_N$, then (3.28) is equivalent to the equality

$$\int (F^{+}\varphi)^{*}\psi dX = \int \varphi^{*}(F\psi)dX.$$

It is easy to see that

$$(\alpha F)^{+} = \alpha^{*}F^{+}, (F_1 + F_2)^{+} = F_1^{+} + F_2^{+}, (F_1F_2)^{+} = F_2^{+}F_1^{+},$$
$$(F^{+})^{+} = F. \tag{3.29}$$

Thus, we see that operation of conjugating is formally applicable to numbers (for which it is reduced to complex conjugation), and to vectors of both types of the formula (3.6), and to the operators. Taking into account this condition and properties of the operators conjugation (3.27), (3.29), we can form the following important rule of transfer from some set correlation to the one conjugated to it:

1) we substitute all numbers for the complex conjugated ones, all

ket vectors for bra vectors, and on the contrary, all operators for conjugated ones;

2) we invert the sequence of vectors and operators.

If $F^+ = F$, then operator F is called a self-conjugate or hermitian. From (3.27) the following main property of the hermitian operator can be found:

$$\langle \varphi | F | \psi \rangle = \langle \psi | F | \varphi \rangle^* . \tag{3.30}$$

Let's remark that hermitian operators are of extreme importance to the VET.

For operator $F = \lambda I$, where I is the unit operator, and λ is the complex number, any vector $|\psi\rangle$ will satisfy to the relation $F|\psi\rangle = \lambda|\psi\rangle$. At the same time, there exist such operators, for which this equality is not true at any vector $|\psi\rangle$ different from zero. For example, the operator in the bi-directional space, which turns all vectors of the plane to the fixed angle with reference to the set center.

Let's introduce the following definition: the vector $|f\rangle \neq |0\rangle$, that satisfies the equation:

$$F|f\rangle = f|f\rangle, \quad \text{or} \quad (F - fI)|f\rangle = |0\rangle, \tag{3.31}$$

is called the eigenvector of the operator F, and the number "f" is the eigenvalue of this operator, which corresponds to this eigenvector. At the same time, according to Dirac, we, for short, designate the eigenvalues and their vectors as one and same letter "f". The vector character of this or that value is expressed by its symbol in brackets - $|\ \rangle$ or $\langle\ |$.

If the given eigenvalue f corresponds to one (to the multiplier) eigenvector $|f\rangle$, it is called simple or nondegenerate. If f corresponds to several linearly independent eigenvectors $|f^{(1)}\rangle, |f^{(2)}\rangle, ..., |f^{(m)}\rangle$, then it is called multiple or degenerated. So, the maximum number m of the indicated vectors is the multiplicity or the degree of degeneration.

It is important to remark that all eigenvectors are determined only to the numerical factor - if $|f\rangle$ is the eigenvector of the operator F with the eigenvalue α, then $\alpha|f\rangle$ at anyone complex α will also be the eigenvector with the same eigenvalue. It is true that:

$$F(\alpha|f\rangle)=\alpha(F|f\rangle|=\alpha(f|f\rangle)=\alpha f|f\rangle=f(\alpha|f\rangle).$$

This arbitrary rule allows us to be limited only by consideration of normalized eigenvectors: if this vector $|f\rangle$ is not normalized, then by multiplying it by the number $\frac{1}{\|f\|}$, we shall transfer it into the vector with the norm is equal to one.

Let's consider important (for the WVET) properties of eigenvalue and eigenvectors of Hermitian operators.

Let's write the equation (3.31) for any two eigenvalues f and f′ of the operator F:

$$F|f\rangle=f|f\rangle,$$
$$F|f'\rangle=f'|f'\rangle,$$

and, using the rule laid down above, we shall pass over from the first equation to one conjugate to it:

$$\langle f|F=f^*\langle f|, \qquad F|f'\rangle=f'|f'\rangle, \tag{3.32}$$

(in the first line, hermicity of the operator has been taken into account: $F^+ = F$). Multiplying the first equation (2.18) at the right by $|f'\rangle$ and the second at the left by $\langle f|$ and making subtraction, we shall receive:

$$0=(f^*-f')\langle f|f'\rangle. \tag{3.33}$$

Having assumed that $|f'\rangle=|f\rangle$, we shall receive:

$$(f^*-f)\langle f|f\rangle=0,$$

wherefore, as $\langle f|f\rangle\neq 0$,

$$f^* = f. \tag{3.34}$$

Thus, all eigenvalues of the Hermitian operator are vital.

By substituting in (3.33) f^* for f′, we shall receive the equality

$$(f-f')\langle f|f'\rangle=0,$$

wherefore

$$\langle f | f' \rangle = 0 \quad \text{at} \quad f \neq f'. \tag{3.35}$$

Thus, eigenvectors of the Hermitian operator, which belong to different eigenvalues, are mutually orthogonal.

Linearly independent eigenvectors, which belong to the degenerated eigenvalue, will not be orthogonal automatically. However, it is always possible to choose such linear combinations, which are already orthogonal one another. Thus, all eigenvectors of the Hermitian operator can be regarded as mutually orthogonal. Besides, according to the above-mentioned, they may be chosen and normalized.

It is proved in linear algebra, any Hermitian operator that is operating in the n-dimensional Euclidean space has exactly the n number of linearly independent vectors. Choosing them as mutually orthogonal and normalized, we may use these vectors for construction of the orthonormalized basis in this space. Its coordinate basis vectors will satisfy the correlation

$$\langle f' | f'' \rangle = \delta_{f'f''} \tag{3.36}$$

and the condition of completeness

$$\sum_{j=1}^{n} | f \rangle \langle f | = I. \tag{3.37}$$

3.1.3. Matrix representation of linear operators

Each matrix of the $n \times n$ order sets some linear operator that is operating in the n-dimensional Euclidean space. Now, we shall prove that, and on the contrary, each linear operator F in the n-dimensional Euclidean space correspond to a certain square matrix of the $n \times n$ order in the given basis.

So, let's presume that in our space there is the orthonormalized basis $|1\rangle, |2\rangle, ..., |n\rangle$, so any vector $|\psi\rangle$ is set by its own components (3.11)

$$\psi_j = \langle j | \psi \rangle. \tag{3.38}$$

Let's work on this vector with the help of the operator F:

$$|\varphi\rangle = F|\psi\rangle.$$

Multiplying both members at the left by $\langle j|$ and using the condition of completeness (3.19), we shall receive:

$$\langle j|\varphi\rangle \equiv \varphi_j = \langle j|F|\psi\rangle = \langle j|FI|\psi\rangle = \sum_{k=1}^{n}\langle j|F|k\rangle\langle k|\psi\rangle \equiv \sum_{k=1}^{n} f_{jk}\psi_k,$$

i.e.

$$\varphi_j = \sum_{k=1}^{n} f_{jk}\psi_k. \tag{3.39}$$

The set of values

$$f_{jk} = \langle j|F|k\rangle \tag{3.40}$$

will form the square matrix - the matrix of the operator F in this basis. This matrix, compared with the operator F, univalently determines components of the vector $|\varphi\rangle = F|\psi\rangle$ by components of the initial vector $|\psi\rangle$, and it is proved by our statement.

Thus, we come to the following important result: the set of all linear operators, which are operating in the n-dimensional Euclidean space, is in the univocal correspondence with the set of all square matrixes of the $n \times n$ order. In other words, each linear operator (by formula (3.40)) is univalently compared in this basis with a certain square matrix, and on the contrary, each square matrix, according to the formula (3.39), is univalently compared with a certain linear operator. This theorem, actually, determines the role played by matrixes in linear algebra and in the WVET.

In particular, using the definition of the unit operator and condition of the basis orthonormality, we shall receive for its matrix the following:

$$(I)_{jk} = \langle j|I|k\rangle = \langle j|k\rangle = \delta_{jk}.$$

Similarly, for the zero operator, we have:

$$(\Theta)_{jk} = \langle j|\Theta|k\rangle = \langle j|\Theta\rangle = 0.$$

So, unit and null operator corresponds to the unit and null matrixes, accordingly. Obviously, the converses are also true.

Let's presume that we have two operators F_1 and F_2. We form operators:

$$F' = F_1 + F_2, F'' = \alpha F_1, F''' = F_1F_2.$$

Using the linearity of the scalar product and the condition of the basis completeness, we shall have:

$$\langle j|F'|k\rangle = \langle j|F_1 + F_2|k\rangle = \langle j|F_1|k\rangle + \langle j|F_2|k\rangle,$$

$$\langle j|F''|k\rangle = \langle j|(\alpha F_1)|k\rangle = \alpha\langle j|F_1|k\rangle,$$

$$\langle j|F'''|k\rangle = \langle j|F_1F_2|k\rangle = \langle j|F_1|F_2|k\rangle = \sum_{l=1}^{n}\langle j|F_1|l\rangle\langle l|F_2|k\rangle,$$

or in matrix symbols:

$$(F_1 + F_2)_{jk} = (F_1)_{jk} + (F_2)_{jk}, \tag{3.41a}$$

$$(\alpha F_1)_{jk} = \alpha(F_1)_{jk}, \tag{3.41b}$$

$$(F_1F_2)_{jk} = \sum_{l=1}^{n}(F_1)_{jl}(F_2)_{lk}. \tag{3.41c}$$

Thus, matrixes of the sum of operators, product of the operator by the number and products of operators are equal, accordingly, to the sum of matrixes of operators, product of the matrix of the operator by the number and to product of matrix of operators.

For matrix of the operator F^+ conjugate to F, we have:

$$(F^+)_{jk} = \langle j|F^+|k\rangle = \langle k|F|j\rangle^* = [(F)_{kj}]^* = [(\tilde{F})_{jk}]^*,$$

where, we used the definition of the matrix of the operator (3.40), the property of the conjugate operator. Thus,

$$(F^+)_{jk} = [(\tilde{F})_{jk}]^*, \tag{3.42}$$

i.e. matrixes of operators F^+ and F are connected by the usual Hermitian conjugate (transposition plus complex conjugation). Now, it is obvious, in particular, that the matrix of the Hermitian operator is the Hermitian one.

Let's formulate the task (3.31) to discover eigenvalues and eigenvectors of the operator F:

$$F|f\rangle = f|f\rangle. \tag{3.43}$$

By multiplying both members at the left by the basis vector $\langle j|$ and by using the definition of the vector components and condition of completeness, we shall receive:

$$\sum_{k=1}^{n} \langle j|F|k\rangle\langle k|f\rangle = f\langle j|f\rangle,$$

i.e.

$$\sum_{k=1}^{n} f_{jk}f_k = ff_j. \tag{3.44}$$

In the expanded aspect

$$\left.\begin{array}{l} f_{11}f_1 + f_{12}f_2 + \ldots + f_{1n}f_n = ff_1, \\ f_{21}f_1 + f_{22}f_2 + \ldots + f_{2n}f_n = ff_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_{n1}f_1 + f_{n2}f_2 + \ldots + f_{nn}f_n = ff_n \end{array}\right\}$$

or

$$\left.\begin{array}{l} (f_{11} - f)f_1 + f_{12}f_2 + \ldots + f_{1n}f_n = 0, \\ f_{21}f_1 + (f_{22} - f)f_2 + \ldots + f_{2n}f_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_{n1}f_1 + f_{n2}f_2 + \ldots + (f_{nn} - f)f_n = 0. \end{array}\right\} \tag{3.45}$$

Thus, in the matrix formalism, the search of eigenvectors of the operator F is reduced to the search of non-trivial solutions of the system of the algebraic linear homogeneous equations (3.8). Nevertheless, such solutions exist only when the determinant of this system is equal to zero:

$$\begin{vmatrix} f_{11} - f & f_{12} & \ldots & f_{1n} \\ f_{21} & f_{22} - f & \ldots & f_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ f_{n1} & f_{n2} & \ldots & f_{nn} - f \end{vmatrix} = 0. \tag{3.46}$$

By opening it, we receive the equation of the n-degree for the eigenvalue of f. By substituting radicals of this equation f into the system (3.45) and solving it, we shall find eigenvectors

$|f\rangle = \{f_1, f_2, ..., f_n\}$ of the operator F, which correspond to eigenvalues of f.

Equation (3.46) is called as the characteristic equation of the operator F, and has n number of radicals. However, some of them may coincide. If the radical of the characteristic equation is not multiple, it is corresponds to one (to the multiple) solution of the system (3.45), i.e. such eigenvalue f is nondegenerate. If the radical is multiple, we shall receive some various solutions of the system (3.8), i.e. such eigenvalue f is degenerated. Thus, the degree of degeneration is equal to the multiplicity of the radical.

As the system of equations (3.45) is homogeneous, then any solution is determined to the numerical multiple. This corresponds to the fact that, if $|f\rangle$ - is the eigenvector of the operator F with eigenvalue f, then $\alpha|f\rangle$ at any complex α will also be the eigenvector with the same eigenvalue.

Let's remark that any linear operator in the n-dimensional Euclidean space has at least one eigenvector. This statement follows from the main theorem of algebra, according to which any algebraic equation of the n-degree (this is the feature of the characteristic equation (3.46)) has at least one (generally, complex) radical.

It was mentioned above that, if it is the Hermitian operator, it has not the one, but n number of linearly independent eigenvectors, which can be used to construct the basis that would satisfying conditions of orthonormality (3.36) and completeness (3.37).

And, at last, we shall find the matrix of the operator F in its own basis, i.e. in the basis of its eigenvectors $|f\rangle$. According to common definition (3.40),

$$(F)_{f'f''} = \langle f' | F | f'' \rangle.$$

Taking into account the equation (3.43) and condition of orthonormality (3.36), we have:

$$(F)_{f'f''} = \langle f' | F | f'' \rangle = \langle f' | f'' | f''' \rangle = f'' \langle f' | f'' \rangle = f'' \delta_{f'f''} = f' \delta_{f'f''},$$

i.e. the matrix of the Hermitian operator F in its own basis is diagonal:

$$(F)_{f'f''} = f' \delta_{f'f''}. \qquad (3.47)$$

Its elements, which are standing on the principal diagonal, are equal to eigenvalues of the operator F (among which, if there is degeneration, there are close ones), and all remaining elements are equal to zero:

$$F = \begin{pmatrix} f_1 & 0 & 0 & \dots & 0 \\ 0 & f_2 & 0 & \dots & 0 \\ 0 & 0 & f_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & f_n \end{pmatrix}.$$

Thus, the purely algebraic problems of diagonalization of matrixes of the given Hermitian operator (i.e. search of basis, in which this matrix is diagonal) is decided simultaneously with the solution of the task of eigenvalues of this operator.

3.2. Axioms of the WVET

Axiom 1. Conditions of the economic system in the WVET are described by vectors $|\psi\rangle$ of the abstract Hilbert space.

It should be remarked that vectors of conditions are not determined identically. They can be multiplied by arbitrary complex number without changes in economic conditions. For description of conditions, we shall choose vectors $|\psi\rangle$, which have the norm equal to one. But, even in this case, there is the possibility of multiplying the vector by the complex number with the only unit, as the norm does not vary at such operation. When solving specific tasks, it is convenient to proceed from some abstract vector $|\psi\rangle$ to a certain set of numbers. It is sufficient to choose the orthonormalized basis from the Hilbert space and to expand the vector $|\psi\rangle$ to its basis vectors:

$$|\psi\rangle = \sum_{j=1}^{\infty} \psi_j |j\rangle. \tag{3.48}$$

Then, it will be unambiguously determined by the set of complex numbers of the ψ_j-components of this vector in the chosen basis:

Axiom 2. The dynamic variable in the WVET are compared with the linear Hermitian operators F operating in the Hilbert space of vectors of

conditions.

As and in case of vectors, to operators in the set orthonormalized basis correspond to the set of numbers

$$f_{ik} = \langle j|F|k \rangle, \qquad (3.49)$$

that form the matrix of the considered operator F. If the operator F transfers the vector $|\psi\rangle$ into the vector $|\varphi\rangle$,its matrix will transform components ψ_j its matrix will transform components φ_j:

$$\varphi_j = \sum_{k=1}^{n} f_{ik} \psi_k .$$

So, conditions in the WVET are described by vectors of the Hilbert space, and dynamic variables – by the linear Hermitian operators that are operating in this space. However, the purpose of the developed WVET is to connect purely mathematical results with results of quantitative observations of economic values, which are used for observation of dynamic variables and some numbers are received - the values of these dynamic variables.

Axiom 3. The only possible result of observation of the economic dynamic variable in the set condition of the system in the WVET are eigenvalues of the operator F, which is compared with it.

So, to find, what values, in principle, can be received at observation of the given dynamic variable, one should solve the task of eigenvalues of its operator:

$$F|f\rangle = f|f\rangle. \qquad (3.50)$$

It pertains to the WVET that the result of observation of the given dynamic variable is usually unpredictable. We cannot exactly say, which of its possible values can be realized at this or that observation, and we can only judged about it with the degree of probability.

Axiom 4. Probability $W_\psi(f)$ of receiving the value of f while measuring the dynamic variable F in the condition $|\psi\rangle$ can be determined by the formula

$$W_\psi(f) = |\langle f|\psi \rangle|^2, \qquad (3.51)$$

where $|f\rangle$ - eigenvector of the operator F belonging to the eigenvalue f.

The function $|\psi\rangle$ - is normalized for the one.

The statement of the axiom 4 admits visual geometrical interpretation. It is possible to construct the orthonormalized basis from all eigenvectors $|f\rangle$ of the Hermitian operator F. In that case, the value $\langle f|\psi\rangle$ will represent the projection of the vector of the condition $|\psi\rangle$ to the basis vector $|f\rangle$. Therefore, the axiom 4 tells us that probability of $W_\psi(f)$ is equal to the square of the unit of the vector $|\psi\rangle$ appropriate projection.

Now, we have the possibility to prove two important theorems referring to the results of measurement by this or that dynamic variable. Let's presume that we have the great many of duplicates of one and the same system, and all system are in one and the same state $|\psi\rangle$. We shall observe (for each such system) a certain dynamic variable F and we shall put a question, what will be the average value of this dynamic variable $\langle F\rangle_\psi$, i.e. its averaged value according to the large number of measurements. The answer to this question is given by the following theorem.

Theorem 1. The average value of dynamic variable F under the condition $|\psi\rangle$ is given by the formula:

$$\langle F\rangle_\psi = \langle\psi|F|\psi\rangle. \tag{3.52}$$

Proof. According to the mathematical statistics, the average value (expectation) of the aleatory variable F is equal to:

$$\langle F\rangle_\psi = \sum_j f\cdot W_\psi(f). \tag{3.53}$$

By using (3.51) and the property of the scalar product hermicity, we shall receive:

$$\langle F\rangle_\psi = \sum_f f|\langle f|\psi\rangle|^2 = \sum_f f\langle f|\psi\rangle^*\langle f|\psi\rangle = \sum_f f\langle f|\psi\rangle\langle f|\psi\rangle. \tag{3.54}$$

Taking into account the equation for the eigenvalue (3.50), it can be written that:

$$\langle F\rangle_\psi = \sum_f \langle\psi|F|f\rangle\langle f|\psi\rangle. \tag{3.55}$$

At last, by using the condition of the basis completeness

$$\sum_f |f\rangle\langle f| = I$$

we shall receive the proved result (3.52) that looks like

$$\langle F\rangle_\psi = \langle f|F|f\rangle = \int \psi^*(x)\cdot \hat{F}\cdot \psi(x)dx.$$

We should remark that

$$\langle F\rangle_\psi = \langle f|F|f\rangle = \langle f|f|f\rangle = f\langle f|f\rangle = f,$$

i.e. the average value of the dynamic variable at the condition that is described by the eigenvector $|f\rangle$ of its operator F, is equal to the eigenvalue f that corresponds to this eigenvector:

$$\langle F\rangle_\psi = f. \tag{3.56}$$

The mean square deviation from its values $\langle F\rangle_\psi$ is used for wide expansion (application) for the purpose of description of the degree of inaccuracy of this or that value.

From this point of view, the following theorem that shows the economic sense of the Hermitian operator eigenvectors is of fundamental importance.

Theorem 2. For the dynamic variable to have a strictly certain value in a certain state, it is necessary and enough that this condition was described by one of eigenvectors $|f\rangle$ of the operator F, compared with the one under review.

Proof. Let's presume that $|\psi\rangle = |f\rangle$, then

$$\langle(\Delta F)\rangle^2_\psi = \langle F^2\rangle_f - \langle F\rangle_f^2 = \langle f|FF|f\rangle - \langle f|F|f\rangle^2 = f^2 - f^2 = 0,$$

i.e. the dynamic variable has a strictly certain value at the condition - $|f\rangle$. Sufficiency is proved.

Now, let's prove the necessity. Let's presume that at certain condition $|\psi\rangle$ the dynamic variable has a certain value, i.e.

$$\langle(\Delta F)^2\rangle_\psi = \langle\psi|\Delta F\Delta F|\psi\rangle = 0,$$

where $\Delta F = F - \langle F\rangle_\psi I$ – the Hermitian operator.

This equality means that the square of the norm of the vector $\Delta F|\psi\rangle$ is equal to zero, and due to the positive certainty of the scalar product, this vector itself is equal to zero:

$$(\Delta F)|\psi\rangle = \left(F - \langle F\rangle_\psi I\right)|\psi\rangle = |0\rangle.$$

Hence

$$F|\psi\rangle = f|\psi\rangle, \quad \text{где } f = \langle F\rangle_\psi,$$

i.e. $|\psi\rangle$ is one of eigenvectors of the operator F.

The theorem is completely proved.

3.3. Correlation and eigenvalues

In the process of the aleatory variable measurement, the observed value will vary from one measurement to another. In the economic - mathematical methods such deviations are frequently measured by the standard deviation (fluctuation)

$$\overline{F} = \overline{(x-\overline{x})^2} = \left[\overline{x^2 - 2x\overline{x} + (\overline{x})^2}\right] = \overline{x^2} - 2\overline{x}\,\overline{x} + (\overline{x})^2 = \overline{x^2} - (\overline{x})^2.$$

It is clear that, if there is no fluctuation, i.e. $x=\overline{x}$ in all measurements, then $\overline{F}=0$. As $(x - \overline{x})^2$ is always the positive value, then it is clear that in all measurements, where x differs from $\overline{x}$, $\overline{F}$ is not equal to zero. The greater the difference between x and $\overline{x}$, the large the $\overline{F}$.

Certainly, it should be remembered that knowledge of $\overline{F}$ and $\overline{x}$ — allows in no way to determine the probability density P(x), but simply provides (generally) the distribution of the value x close to its average value. Indeed, we can say that the value $\overline{(x-\overline{x})^2}$ is the criterion of uncertainty x, as it shows approximately how the values of x will vary from one measurement to another. Therefore, we can lay it down as $\overline{(x-\overline{x})^2} = (\Delta x)^2$, where Δx is the uncertainty x.

Generalization in case of the WVET. These considerations are easy to generalize in case of the WVET. If wave function ψ(x) is known, then the probability density is also known:

$$P(x) = \psi^*(x)\,\psi(x),$$

and consequently we know the average value of any x-function. In particular, the average value F is given by the expression

$$F = \int_{-\infty}^{\infty} \psi^*(x)\,(x-\bar{x})^2\,\psi(x)dx = (\Delta x)^2. \tag{3.58}$$

However, very often, it is inconvenient to deal with the detailed form of the wave function, as this requires the solution of the wave equation. Sometimes, it is enough to know only the general character of distribution, namely, the value $\bar{F}$ and $\bar{x}$, which, in general, characterize the main common properties of distribution. In particular, later, we shall see that it is possible to draw certain conclusions on the value $(x-\bar{x})^2$, even when the exact form of the function ψ is unknown. Introduction of the averages of $\bar{F}$ type seems to be rather useful.

Correlation between $\dot{X}$ and X. At any statistical distribution of two economic variables, such as $\dot{X}$ and X, it is important to know, whether these two variables correlate or not. For example, among people, there is no simple relation between their growth and weight, but these two values are statistically connected (correlated), as usually the taller person is heavier, than the shorter one. Similarly, one may ask, whether there is any connection (correlation) between distribution $\dot{X}$ and distribution X? In other words, whether the greater $\dot{X}$ usually corresponds to the greater X or, on the contrary, the greater $\dot{X}$ is met at the smaller X? If one of these statistical connections exists, it is possible to tell that $\dot{X}$ and X correlate. On the other hand, if there is no such connection, then one can say about these two values that they are statistically independent.

Let's presume that the height of people h and their weight w are statistically independent. This means that distribution of height does not depend on the weight. Then, it can be written down that the probability of having the given height between h and h+dh is equal to R(h)dh. Similarly, the probability of having any weight between w and w+dw does not depend on h and, therefore, it is equal to S(w)dw. By definition, the probability of the two independent results together is equal to the product of their probabilities. Therefore, the probability of that the height of the person is between h and h +dh, and the weight is between w and w +dw is equal to the product:

$$\psi(h, w)dh\,dw = R(h)\,S(w)\,dh\,dw.$$

If the distribution cannot be written down, as the product, then the two variables are not independent statistically. Let's consider, for example, the following formula: P (h, w) = 1 / ($h^2 + w^2$). It is clearly that the distribution function, with reference to h cannot be considered as independent on w.

Quantitative measure of correlation. The following average value is a convenient quantitative measure of correlation degree of two economic values:

$$C_{11} = \overline{\left(X - \overline{X}\right)\left(\dot{X} - \overline{\dot{X}}\right)} = \overline{X\dot{X}} - \overline{X}\,\overline{\dot{X}}. \tag{3.59}$$

If distribution X is statistically independent on distributions $\dot{X}$, then C_{11} = 0, as in this case:

$$\overline{X\dot{X}} = \int R(X)\,S(\dot{X})\,X\dot{X}\,dX\,d\dot{X} = \overline{X}\,\overline{\dot{X}}.$$

However, it is possible that C_{11} = 0, even if there is correlation. For example, the larger values of |X| may correlate with the larger values of $|\dot{X}|$, but they correlate in such a way that for each value of X the value of $\dot{X}$ (with equal probability) may be both positive and negative. Therefore, both $\overline{X\dot{X}}$ and $\overline{X}\,\overline{\dot{X}}$ disappear, even if there is any correlation.

The degree of correlations for this more complicated case can be received by considering the function:

$$C_{22} = \overline{X^2\dot{X}^2} - \left(\overline{X}\right)^2\left(\overline{\dot{X}}\right)^2. \tag{3.60}$$

It is clear that the value C_{22} is equal to zero, if X and $\dot{X}$ are statistically independent, but is not equal to zero in the above case, when C_{11} = 0. However, generally, there may exists even more complicated cases of correlation, in which both C_{11} and C_{22} disappear. To cover all possible cases of correlation, it is necessary to study the function of the following type:

$$C_{nm} = \overline{X^n\dot{X}^m} - \left(\overline{X}\right)^n\left(\overline{\dot{X}}\right)^m. \tag{3.61}$$

Classification of classical statistical systems by the average values

of the product $X^n\dot{X}^m$. The above consideration shows the significance of the averages of all members of the $X^n\dot{X}^m$ type. Members of the type $\overline{X^n}$ and $\overline{\dot{X}^m}$ may be considered similarly to the members $\overline{X}, \overline{X^2}, \overline{\dot{X}}$ and $\overline{\dot{X}^2}$, but only if used for measurements of more complicated and subtle properties of fluctuation. Therefore, owing to complete information about fluctuations and correlations, it is possible to calculate all values of $X^n\dot{X}^m$. Using the product of $X^n\dot{X}^m$, one can draw the arbitrary function $f(X, \dot{X})$. Its average value is equal to:

$$\bar{f}(X,\dot{X}) = \sum \overline{A_{nm} X^n \dot{X}^m} = \sum A_{nm} \overline{X^n \dot{X}^m}.$$

This means that fluctuations and correlations determine the average value of any economic value and they describe all features of distribution we are interested in. In statistics, $X^n\dot{X}^m$ is called n, m-moment of distribution by analogy with the angular momentum in mechanics.

If there are no fluctuations, we should have $\bar{f}(X,\dot{X}) = f(\overline{X}, \overline{\dot{X}})$. These two values differ only because of fluctuation. Each form of the function is sensitive to certain types of fluctuations and correlations, which depend on the size of the coefficient for each member $X^n\dot{X}^m$ in the power series of this function expansion.

In the WVET, correlation functions are received by simple replacement of X for the operator $\hat{\dot{X}}$, and the product $X\dot{X}$ - by the average of two products with the transposed multiplicands $\frac{1}{2}(X\dot{X} + \dot{X}X)$. Thus,

$$C_{nm} = \frac{1}{2}\int \psi^* \left[X^n\dot{X}^m + \dot{X}^m X^n\right]\psi dx - \left(\int \psi^* X^n \psi dX\right)\left[\int \psi^* \dot{X}^m \psi dX\right] \quad (3.62)$$

3.4. Evolutionary equation

To complete construction of the WVET, it is necessary:

1) to compare each observed economic value with a certain Hermitian operator that operates in the Hilbert space of

condition vectors;

Operator	Value
Profitability $\hat{X}$	X
PRC $\hat{\dot{X}}$	$\dot{X}$
Risk $\hat{R}$	$\dot{X}^2$

2) to find the type of dynamic equations, which define the development of the given economic system in time.

To receive the equation that describes the evolution of the economic system status, we should use the method of propagator. Main object of this approach is propagator K $(q, t; q_0, t_0)$, which allows to express the wave function $\psi(q, t)$ through its initial value $\psi(q_0, t_0)$ at the point of time $t = t_0$.

Here, q is any profitability that describes our system in point of time t, and q_0 are the same variables at the point of time t_0. In these symbols, the propagator K is determined by the relation:

$$\psi(q, t)=\int K(q, t; q_0, t_0)\, \psi(q_0, t_0)dq_0. \qquad (3.63)$$

Let's consider first the main properties of the operator K. Let's presume that at the point of time $t = t_0$ profitabilities q had one definite value $q = q_0$. In this case $\psi(q'_0, t_0) = \delta(q'_0 - t_0)$. If at the point of time t $q=q'$, then according to (11), we receive:

$$\psi(q', t) = \int K(q', t; q_0, t_0).$$

Hence, the value

$$P(q', t; q_0, t_0)=|\psi (q', t)|^2=|K(q', t; q_0, t_0)|^2$$

is the probability of the system transition from the state $q=q_0$ into the state $q=q'$ for the time $t - t_0$ $(t_0 < t)$. Propagator K has an important property: the product of propagators is a propagator. Really, if we take the function $\psi(q', t)$ as the initial one and insert it into (3.63), then we shall receive:

$$K(q, t; q_0, t_0) = \int K(q, t; q'', t'')\, K(q'', t''; q_0, t_0)dq''. \qquad (3.64)$$

From (3.64) it is clear that transition of the system from the state q_0 in which it was at the point of time t_0, into the state q by the moment of time t $(t > t_0)$ can be considered in two stages. At the beginning, the system go over to any intermediate state q″ at the point of time t″ ($t_0 < t'' < t$), and only after that transition it goes over to the final state q by the moment of time t.

It is obvious that it is possible to continue dividing the interval (t, t_0). Let's divide it into N intervals: (t_0, t_1), (t_1, t_2)..., (t_k, t_{k+1}), …, (t_{N-1}, t_N), $t_N = t$. Values of the dynamic variables in the given moments of time should be designate by us as q_k (k = 0, 1, …, N), so that the propagator K that refers to *l*-interval, would have the following form:

$$K_l = K(q_{l+1}, t_{l+1}; q_l, t_l).$$

By successively applying the K_l to any initial function $\psi(q_0, t_0)$, we shall receive the following expression of propagator for the period of time (t_0, t)

$$K(q, t; q_0, t_0) = \int\ldots\int K(q, t; q_{N-1}, t_{N-1})\, K(q_{N-1}, t_{N-1}; q_{N-2}, t_{N-2})\ldots$$
$$\ldots K(q_2, t_2; q_1, t_1)\, K(q_1, t_1; q_0, t_0) dq_{N-1}, dq_{N-2} \ldots dq_1, \quad (3.65)$$

where Integration is carried in all intermediate states (integral of multiplicity N – 1).

Process of successive transition through all admissible intermediate states is called as Markov's chain. However, in the classical theory, this chain will be formed not by the amplitudes of transition, but by probabilities of transition $P(q_{k+1}, t_{k+1}; q_k, t_k)$:

$$P(q, t; q_0, t_0) = \int\ldots\int P(q, t; q_{N-1}, t_{N-1})\, P(q_{N-1}, t_{N-1}; q_{N-2}, t_{N-2})\ldots$$
$$\ldots P(q_2, t_2; q_1, t_1)\, P(q_1, t_1; q_0, t_0) dq_{N-1}, dq_{N-2} \ldots dq_1. \quad (3.65')$$

Figure 21 shows several "trajectories", which appear in Markov's chain. We have put the word "trajectories" in quotation marks, as any final time interval $\Delta t = t_{k+1} - t_k$ can be divided into smaller intervals $\Delta t' << \Delta t$. In their turn, these intervals can be further divided in such a way that the trajectories in Markov's chain have no continuous tangents.

Let's remark that chains (3.65) and (3.65′) differ by the fact that in the WVET amplitudes of probability instead of probabilities are of fundamental significance.

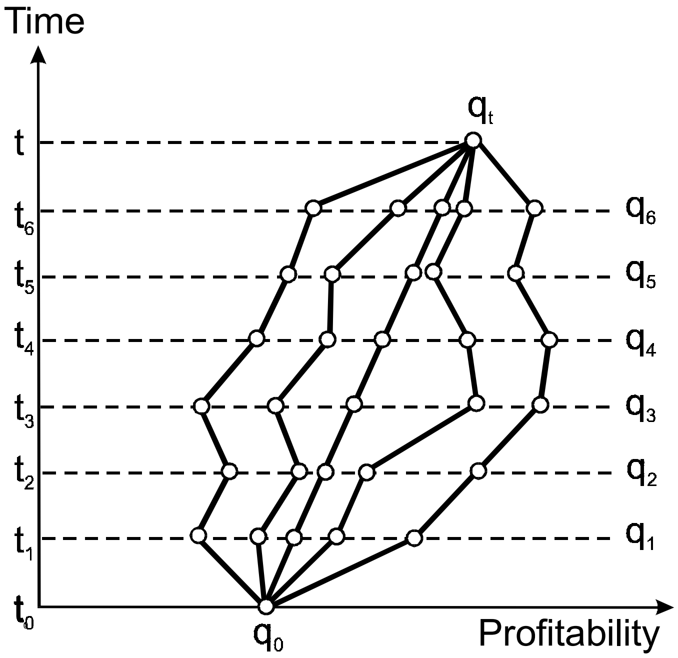


Fig.21. Trajectories of profitability change used for integration in Markov's chain

To find the clear expression for the propagator (q, t; q_0, t_0), we shall consider the special case: behaviour of the independent single investor.

In this case, the function of consumer choice (FCC) has the following form:

$$L(X, \dot{X}) = \dot{X}^2 - U(x) = \hat{R} - U(x),$$

where X – profitability,

$\dot{X}$ – rate of the change of profitability,

$\dot{X}^2$ – risky item,

U(x) – utility function,

$\hat{R}$ – operator of the risk.

Operation S within the short time interval (t_k, t_{k+1}) is equal to:

$$S(X_{k+1}, t_{k+1}; X_k, t_k) = \int_{t_k}^{t_{k+1}} L(X, \dot{X}) dt.$$

Let's show now that, if the propagator K for the infinitely short time interval $\Delta t = (t_k - t_{k+1})$ has the following form:

$$K\left(X_{k+1}, t_{k+1}; X_k, t_k\right) = C \cdot \exp\left\{ \frac{i}{h} \left[\left(\frac{X_{k+1} - X_k}{\Delta t} \right)^2 - V\left(X_k\right) \right] \Delta t \right\}, \quad (3.66)$$

where C and h – certain constants,

i - imaginary unit, $i = \sqrt{-1}$,

then the wave function ψ(x, t) that is determined by the formula (1), will satisfy the evolutionary equation:

$$ih \frac{\partial \psi(X, t)}{\partial t} = -\frac{\pi^2}{4} D^2 \psi(X, t) + V(x) \cdot \psi(X, t). \quad (3.67)$$

Let's remark that the value $\frac{X_{k+1} - X_k}{\Delta t}$ approximates the profitability rate of change at the interval of time (t_k, t_{k+1}), and C is the normalizing factor that is determined from the condition $K = \delta(X_{k+1} - X_k)$ at $\Delta t \to 0$. It is easy to find that:

$$C = \left(\frac{1}{\pi i h \Delta t} \right)^{1/2}. \quad (3.68)$$

Now, substitute (3.66) in (3.63) and presume that $q_0 = X - \xi$, $q - q_0 = X - X_0 = \xi$, $t = t_0 + \Delta t$. Further

$$\psi(x_0, t_0) = \psi(x - \xi, t_0) = \psi(x, t_0) - \frac{\partial \psi(x, t_0)}{\partial x} \xi + \frac{1}{2} \frac{\partial^2 \psi(x, t_0)}{\partial x^2} \xi^2 + \ldots$$

and

$$\exp\left\{ -\frac{i}{h} V(x) \Delta t \right\} = 1 + \frac{1}{ih} V(x) \Delta t + \ldots$$

Expression (11) now has the following form:

$$\psi(x, t_0 + \Delta t) = C \int_{-\infty}^{+\infty} d\xi \exp\left(\frac{i}{h \cdot \Delta t} \xi^2 \right) \left[1 + \frac{1}{ih} V(x) \cdot \Delta t + \ldots \right] \times$$
$$\times \left[\psi(x, t_0) - \frac{\partial \psi(x, t_0)}{\partial x} \xi + \frac{1}{2} \frac{\partial^2 \psi(x, t_0)}{\partial x^2} \xi^2 + \ldots \right]. \quad (3.69)$$

By using $\int\limits_{-\infty}^{+\infty} e^{iaz^2} = \sqrt{\frac{a\pi}{2}}$, it is easy to calculate the right member of the formula (3.69). The integral that contains the factor $\psi(x, t_0)$, due to normalization (3.68) is equal to 1. Integration of the item, which is linear by ξ, gives zero. The integral that contains ξ^2, is equal to $-\frac{1}{ih}\frac{h^2}{4}\Delta t$. Members of the higher degree by ξ tend to zero faster, than $(\Delta t)^{3/2}$. By collecting the results of Integration and noticing that $\frac{1}{\Delta t}[\psi(x, t_0 + \Delta t) - \psi(x, t_0)] \to \frac{\partial \psi(x, t)}{\partial t}$ (we have substituted t_0 for t, as $t \to t_0$ at $\Delta t \to 0$), we receive the evolutionary equation (3.67) for the wave function $\psi(x, t)$, which was determined with the help of (3.63) and (3.66).

Following the above, it is possible to determine the propagator for the final time interval (t_0, t). By multiplying propagators (3.66) for the intermediate intervals (t_k, t_{k+1}) and integrating according to the intermediate values of the variables x_k, we shall find:

$$K(x, t; x_0, t_0) = \lim_{\substack{N\to\infty \\ \Delta t \to 0}} \int \ldots \int \exp\left\{\frac{i}{h}\sum_{k=1}^{N-1}\left[\left(\frac{x_{k+1} - x_k}{\Delta t}\right)^2 - V(x_k)\Delta t\right]\right\} \times$$

$$\times C^{\frac{N}{2}} dx_1\, dx_2 \ldots dx_{N-1}. \tag{3.70}$$

This limit of the multiple integral is called the functional interval, noting that at the indefinitely subtle separation of the interval (t_0, t) the value $\frac{x_{k+1} - x_k}{\Delta t}$ may be treated as the rate $\frac{dx}{dt} = \dot{x}$, and, by designating the differential of volume of Integration $C^{\frac{N}{2}} dx_1\, dx_2 \ldots dx_{N-1}$ as $d\{x\}$, we may write down the result (3.70) in the short form

$$K(x, t; x_0, t_0) = \int d\{x\} \exp\left\{\frac{i}{h}\int\limits_{t_0}^{t} L(x, \dot{x})dt\right\}. \tag{3.71}$$

The integral in the exponent is the classical operation

$$S=\int_{t_0}^{t} L(x,\dot{x})dt. \qquad (3.72)$$

Integration in the formula (3.71) is applied not only to classical trajectories, which correspond to the extremum of the integral (3.72), also to all trajectories that are connecting the points (t_0, x_0) and (t, x).

As it has been mentioned above, in the WVET, the causality principle is laid down as the evolutionary equation:

$$ih\frac{\partial\psi(t)}{\partial t}=H\psi(t), \qquad (3.73)$$

the solution of which allows to determine the wave function in arbitrary moment of time t, if it is known, say, at the point of time t = 0 in the example of the securities with profitability x. Let's presume that its wave function is ψ(x). We shall presume that ψ(x) does not belong to neither the operator of profitability $\hat{X}$, nor to the conjugated operator of the profitability rate of change $\hat{\dot{X}}$. Making observation of the values of profitability and profitability rates of change, it is possible to determine the probability density of profitability values $|\psi(x)|^2$, and also the probability density of the profitability rate of change values $|C(p)|^2$.

The indicated densities of probabilities constitute the maximum of information that can be received on the state of profitability of the securities.

As the complete description of profitability of the securities in the WVET is carried out by setting the function ψ(x), then this statistical information should be sufficient for determination of ψ(x). According to the principle of superposition of the state and the fact that $\hat{X}$ and $\hat{\dot{X}}$ constitute the pair of canonically conjugate variables,

$$\psi(x)=\frac{1}{\sqrt{2\pi h}}\int_{-\infty}^{\infty} dp\, c(p)\, e^{\frac{i}{h}px}. \qquad (3.74)$$

Supposing that

$$\psi(x)=|\psi(x)|e^{i\alpha(x)},\ c(p)=|c(p)|e^{i\beta(p)}, \tag{3.75}$$

where α(x) and β(p) — are the real functions of their arguments, we receive, by substituting (3.75) in (3.74) and dividing real and imaginary members:

$$|\psi(x)|\cos\alpha(x)=\frac{1}{\sqrt{2\pi h}}\int_{-\infty}^{\infty} dp\, c(p)\cos\left(\frac{px}{h}+\beta(p)\right),$$
$$|\psi(x)|\sin\alpha(x)=\frac{1}{\sqrt{2\pi h}}\int_{-\infty}^{\infty} dp\, c(p)\sin\left(\frac{px}{h}+\beta(p)\right).$$

(3.76)

As it is seen from (3.76), the task of the wave function definition by "experimental" data admits exact definition: it is necessary to solve the system of two integral equations (3.76) with reference to the two unknown of real functions α(x) и β(p), and by considering as the known $|\psi(x)|$ and $|c(p)|$.

Equations (3.76) are rather complicated from the mathematical point of view. For example, setting of limitations, which should satisfy the functions $|\psi(x)|$ and $|c(p)|$ for solving of these equations, is the nontrivial problem. Therefore, we shall consider only the example, for which it is possible to find the obvious solution of equations (3.76). Let's presume that distribution of profitability and PRC (for simplification) is governed by the normal law:

$$|\psi(x)|=\sqrt[4]{\frac{2}{\pi a^2}}\exp\left\{-\frac{(x-x_0)^2}{2a^2}\right\},$$
$$|c(p)|=\sqrt[4]{\frac{2ka^2}{\pi h^2}}\exp\left\{-\frac{ka^2(p-p_0)^2}{2h^2}\right\},\quad k>0,$$
$$\int_{-\infty}^{\infty}dx\,|\psi(x)|^2=\int_{-\infty}^{\infty}dp\,|c(p)|^2=1.$$

(3.77)

where x_0 and p_0 - average values of profitability and PRC, accordingly;

a and k – parameters, which define the degree of localization.

Representing $\psi(x)$ и $c(p)$ in the form of (3.75) and by using the Fourier transform for phase functions, we find that:

$$\psi_{\pm}(x)=\left(\pi a^2\right)^{-1/4}\left(\sqrt{k}\mp i\sqrt{1-k}\right)^{1/2}\exp\left\{-\frac{(x-x_0)^2}{2a^2}\pm i\frac{(x-x_0)^2}{2a^2}\times\right.$$
$$\left.\times\sqrt{\frac{1-k}{k}}\pm\frac{i}{h}p_0x+i\gamma\right\},$$

$$c_{\pm}(p)=\left(\frac{ka^2}{\pi h^2}\right)^{1/4}\exp\left\{-\frac{ka^2(p-p_0)^2}{2h^2}\mp i\frac{a^2(p-p_0)^2}{2h^2}\times\right.$$
$$\left.\times\sqrt{k(1-k)}\mp\frac{i}{h}x_0(p-p_0)+i\gamma\right\}. \qquad (3.78)$$

First of all, we shall our pay attention to the fact that phase functions are equal

$$\alpha_{\pm}(x)=\pm\frac{(x-x_0)^2}{2a^2}\sqrt{\frac{1-k}{k}}\pm\frac{p_0x}{h}+\arg\left(\sqrt{k}\mp i\sqrt{1-k}\right)^{1/2}+\gamma,$$
$$\beta_{\pm}(p)=\mp\frac{a^2(p-p_0)^2}{2a^2}\sqrt{k(1-k)}\mp\frac{1}{h}x_o(p-p_0)+\gamma. \qquad (3.79)$$

3.5. Variational method of determination of the investment portfolio risk

3.5.1. Variational form of the task on eigenvalues of the investment portfolio risk

To definition of bound states by the variational method, the WVET uses the functional. At the beginning, we shall prove the following theorem:

Theorem. Let's presume that R is the risk operator of the investment

portfolio and N [Ψ] is the average risk value of the portfolio:

$$N[\Psi] \equiv \frac{\langle \Psi | R | \Psi \rangle}{\langle \Psi | \Psi \rangle}. \tag{3.80}$$

Any eigenvector, for which the average risk value of the portfolio is permanent, is the eigenvector of the discrete spectrum of the operator R. The inverse is also true. The appropriate eigenvalue is equal to the stationary value of the functional N [Ψ].

It should be remarked that we speak about vectors with the final norm: function space F is the Hilbert space of dynamic states of the economic system. Therefore, the theorem assert that eigenfunctions R, which belong to the Hilbert space, are solutions of the variational equation

$$\delta N = 0. \tag{3.81}$$

Let's also note that the functional N[Ψ] does not depend on the norm and the vector phase $|\Psi\rangle$, so, the theorem remain true, if some additional condition are applied to these values. In particular, it is sometimes convenient to limit the area of change $|\Psi\rangle$ by vectors with the unit norm, as it has been made in some examples of this chapter.

Proof of the theorem. Let's calculate the variation N[Ψ]

$$\langle \Psi | \Psi \rangle \delta N = \delta(\langle \Psi | R | \Psi \rangle) - N\delta(\langle \Psi | \Psi \rangle) =$$
$$= \langle \delta\Psi | (R - N) | \Psi \rangle + \langle \Psi | (R - N) | \delta\Psi \rangle.$$

As the value $\langle \Psi | \Psi \rangle$ remains final and is not equal to zero, the equation (3.80) is equivalent to the following:

$$\langle \delta\Psi | (R - N) | \Psi \rangle + \langle \Psi | (R - N) | \delta\Psi \rangle = 0. \tag{3.82}$$

The vector $|\delta\Psi\rangle$ is the variation of the vector $|\Psi\rangle$, and $\langle \delta\Psi|$ is the variation of the vector conjugate to $|\Psi\rangle$. Therefore, variations $|\delta\Psi\rangle$ and $\langle \delta\Psi|$ are not independent. They can be, however, considered as such. Really, by substituting $|\delta\Psi\rangle$ in the equation (3.82), which is true for any infinitesimals $|\delta\Psi\rangle$, for $i|\delta\Psi\rangle$

$$-i\langle\delta\Psi|(R-N)\Psi\rangle+i\langle\Psi|(R-N)\delta\Psi\rangle=0, \tag{3.82′}$$

and by forming the appropriate linear combinations of the equations (3.82) and (3.82′), we shall receive two equivalent equations:

$$\langle\delta\Psi|(R-N)\Psi\rangle=0 \quad \text{и} \quad \langle\Psi|(R-N)\delta\Psi\rangle=0.$$

They are equivalent to the equation (3.82), if we presume that variations $|\delta\Psi\rangle$ and $\langle\delta\Psi|$ are arbitrary and independent.

We received two equations:

$$(R-N)|\Psi\rangle=0, \quad \langle\Psi|(R-N)=0,$$

or

$$(R-N[\Psi])|\Psi\rangle=0, \tag{3.83a}$$

$$(R^{+}-N^{*}[\Psi])|\Psi\rangle=0. \tag{3.83b}$$

Due to hermicity H ($R = R^{+}$), equations (3.83a) and (3.83b) are identical. Therefore, the equation (3.81) is equivalent to the equation (3.83a): any vector $|\Psi_1\rangle$, for which the functional N is stationary, is the eigenvector R with eigenvalue $N[\Psi_1]$.

And inversely, let's presume that $|\Psi_1\rangle$ — is the eigenvector with the final norm and R_1 is the appropriate eigenvalue

$$R|\Psi_1\rangle=R_1|\Psi_1\rangle.$$

By multiplying this equation at the left by $\langle\Psi_1|$, we receive

$$R_1 = R[\Psi_1].$$

Therefore, the vector $|\Psi_1\rangle$ satisfies the equation (3.83) due to the hermicity R and reality N_1 and to the equation (3.83b). From here, we conclude that the functional $N[\Psi]$ is stationary for $\Psi = \Psi_1$.

We have seen earlier that the approximate solution of the variational equation (3.81) can be received by limiting the area of vectors $|\Psi\rangle$ only by the part of the state space. At successful choice of this area F′, we receive some eigenvectors R with high accuracy, and their eigenvalues – with even higher accuracy.

The method becomes especially simple in case, when the trial function is linearly dependent on variational parameters i.e., when F′ is also the vector space. Then, F′ is the subspace F in the ordinary sense.

Let's introduce the following symbols: P - projector on F′, Φ - arbitrary vector F′, and R_P – reduction of the risk operator by F′

$$H_P \equiv PHP^{-1}. \tag{3.84}$$

The functional N [Φ] (3.80) is equal to the average value R_P. The Hermitian operator R_P is linearly transforming the vectors from F′ to itself and may be considered as the Hermitian operator in the space F′, for which the theorem is true. Therefore, the variational equation

$$\delta N[\Phi] = 0 \tag{3.87}$$

is equivalent to the equation for the eigenvalues

$$R_P\Phi = N\Phi. \tag{3.88}$$

Thus, variational approximation includes replacement of the task on the eigenvalues of the operator R by the similar task that is *a priori* easier to solve, as it is defined in a narrower space.

3.5.2. Computation of the risk investment portfolio

We have the equality:

$$\hat{R}\cdot\Psi = N\Psi. \tag{3.89}$$

Let's write this the equation in R-representation. For this purpose, first, divided f(x) into the Fourier series according to the eigenfunctions of the operator $\hat{R}$:

$$f(x) = \sum_n c_n \psi_n(x).$$

Then, after substituting f(x) into the equation (1), we shall receive:

$$\hat{R}\cdot\sum_n c_n \psi_n(x) = N\sum_n c_n \psi_n(x). \tag{3.90}$$

By multiplying scalarly by ψ_m (x), we receive:

$$\sum_n \int \psi_m^*(x)\cdot \hat{R}\cdot \psi_n(x)\cdot c_n \cdot dx = N\int c_m \cdot \psi_m^*(x)\cdot \psi_n(x)\cdot c_n \cdot dx,$$

$$\sum_n R_{mn} \cdot c_n = N \cdot c_m. \tag{3.91}$$

The equation (3.91) represents the finite or infinite system of the linear uniform algebraic equations (depending on the number of eigenvalues of the operator $\hat{R}$). N and c_n are unknown. For this system to receive the non-trivial solution, the determinant composed of coefficients at the unknown c_n, should be equal to zero:

$$\begin{vmatrix} R_{11} - N & R_{12} & R_{13} & \cdots & \cdots \\ R_{21} & R_{22} - N & R_{23} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{i1} & R_{i2} & \cdots & R_{ii} - N & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{3.92}$$

Roots of the equation (3.92) give the spectrum of values N_n. Then, by substituting the determined value N_n into the system (3.91), we find c_n.

Thus, the solution of the differential equation of risk $\hat{R}\cdot \Psi = N\Psi$ can lead to the solution of the finite or infinite system of linear uniform algebraic equations.

Each value of N_i^i corresponds to a certain combination of securities into the portfolio

$$f(x) = \sum_n c_n^i \psi_n.$$

And $(c_n^i)^2$ is equal to probability of finding this asset in the portfolio, and N^i corresponds to the value of risk of the i-portfolio.

Example

Basic data: Probability density of distribution of profitability and profitability rate of change of nine assets.

Task: To find the portfolio with the minimum risk.

Assumptions: Distribution of profitability and profitability rate of change follows normal distribution.

Basic data in graphical form look like (fig. 22-74):

1-st assets:

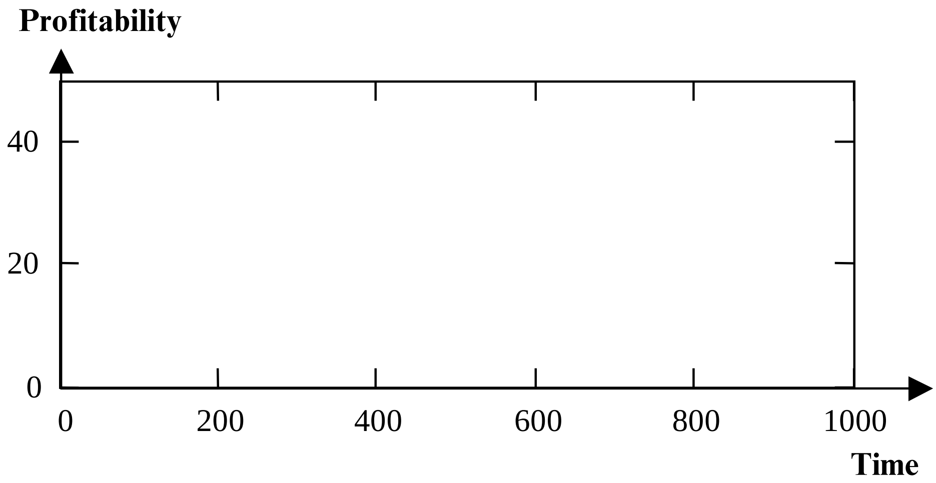


Fig. 22. Dependence of profitability of the 1-st asset on time

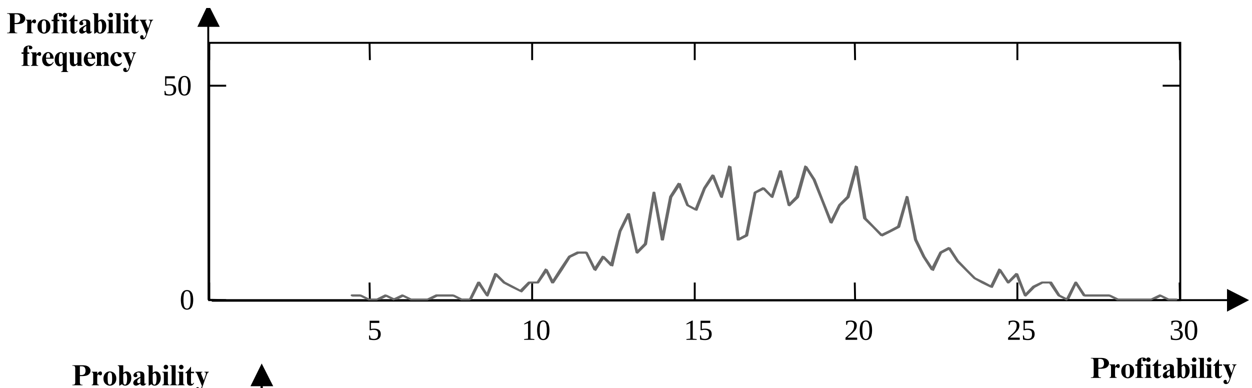
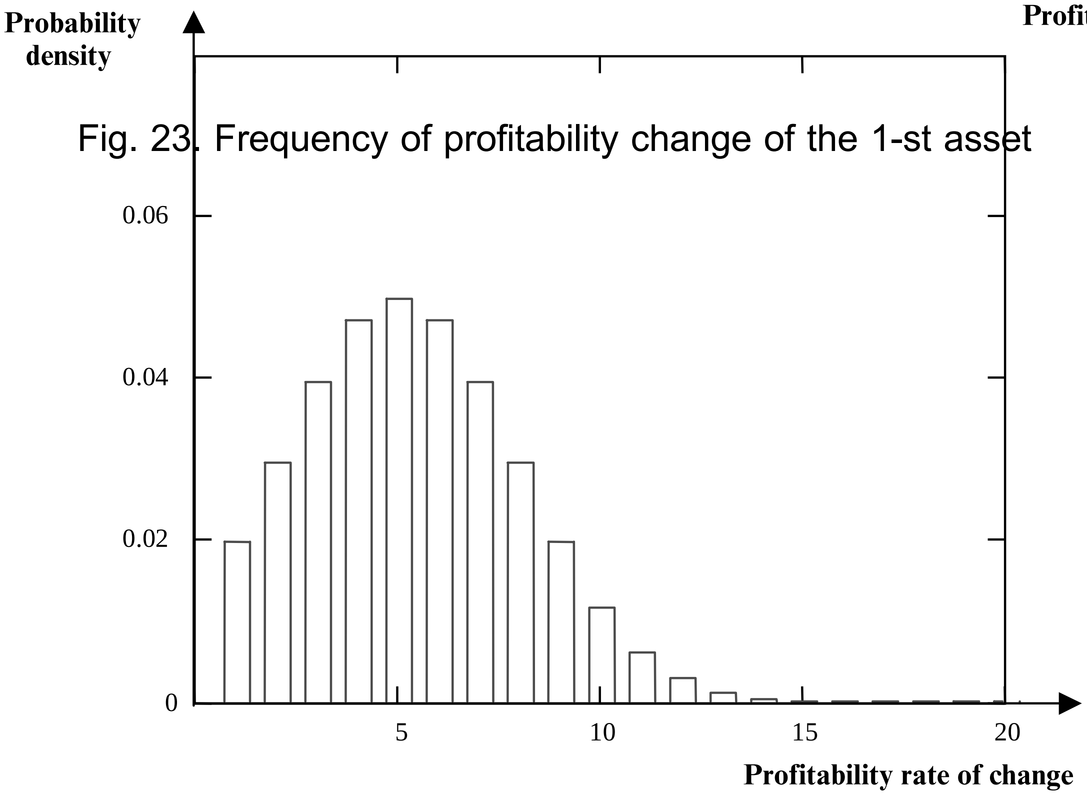


Fig. 23. Frequency of profitability change of the 1-st asset



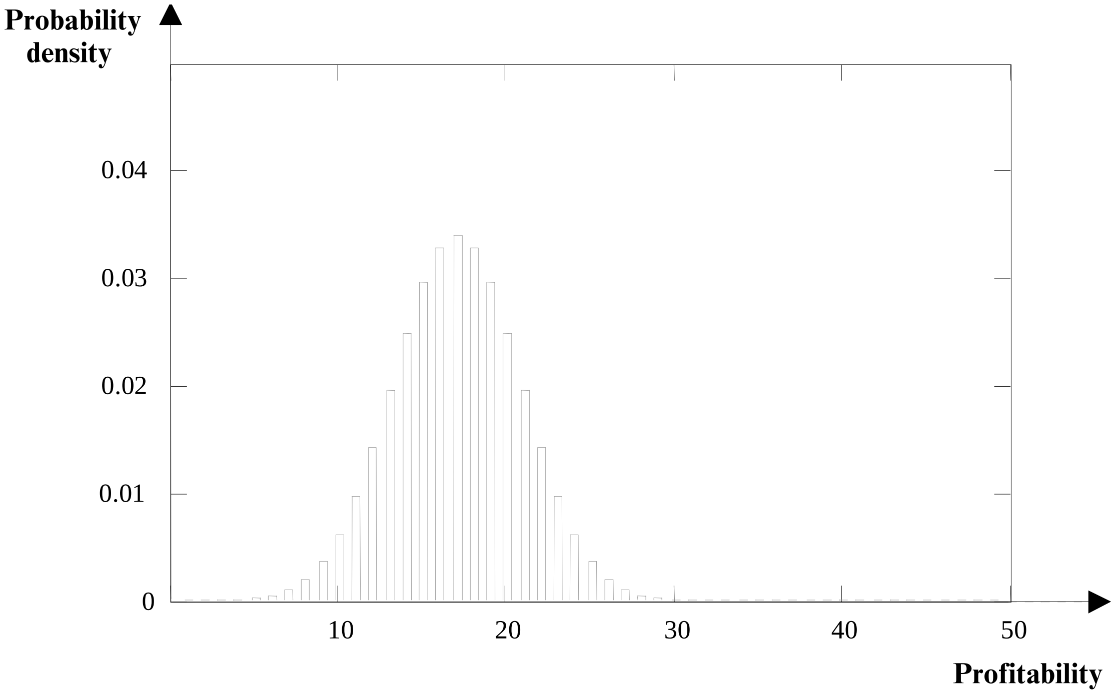


Fig. 24. Probability density of profitability of the 1-st asset

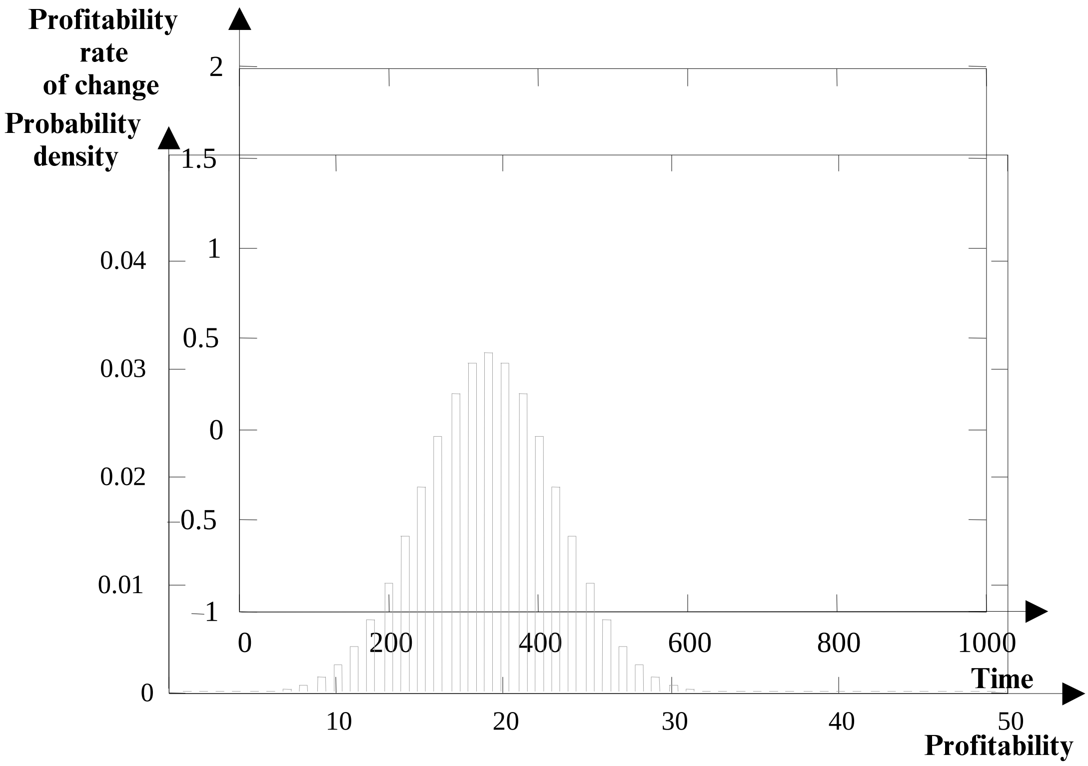


Fig. 25. Profitability rate of change of the 1-st asset

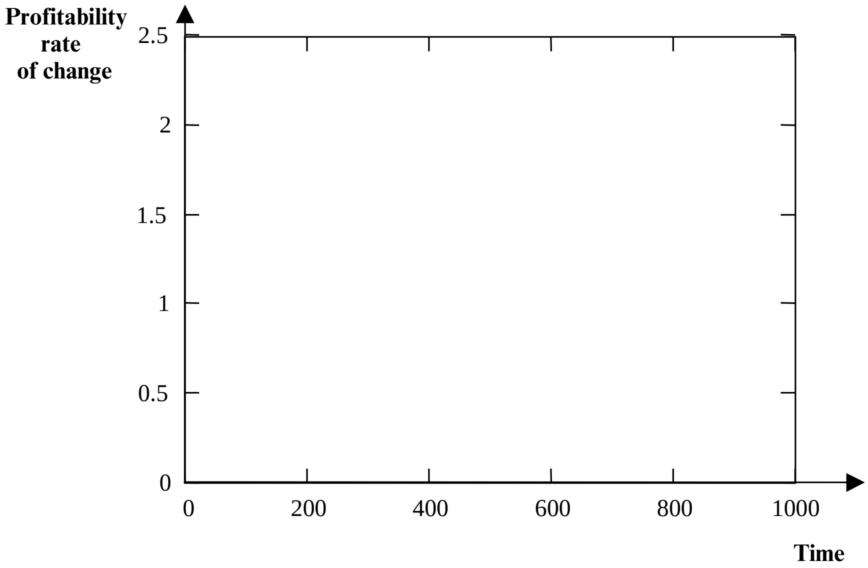


Fig. 31. Profitability rate of change of the 2-nd asset

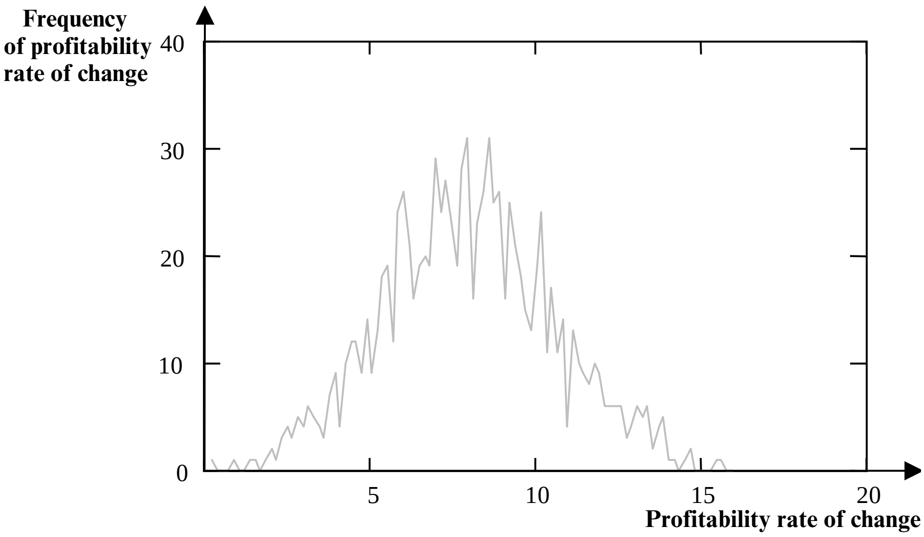


Fig. 32. Frequency of profitability rate of change of the 2-nd asset

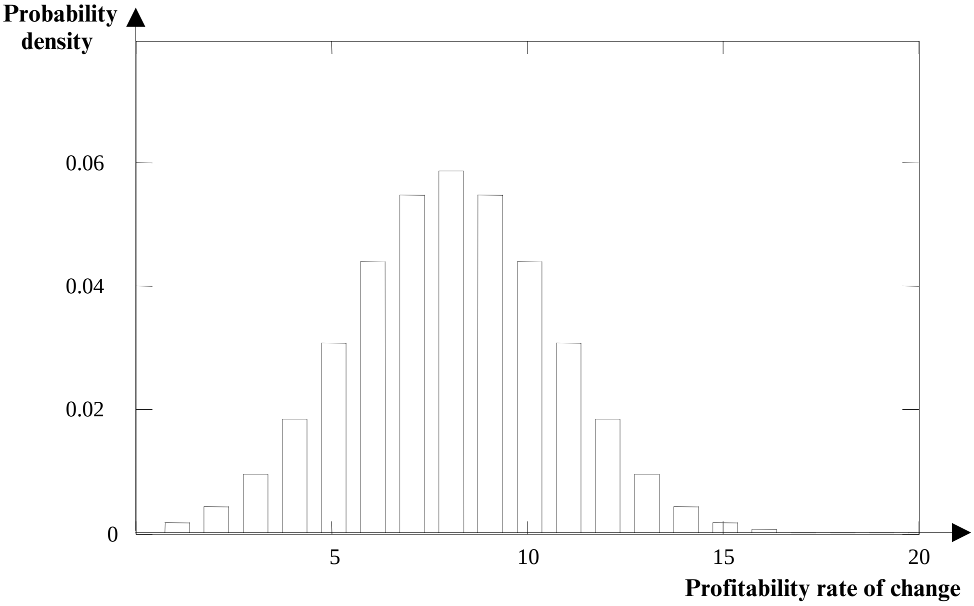


Fig. 33. Probability density of profitability rate of change of the 2-nd asset

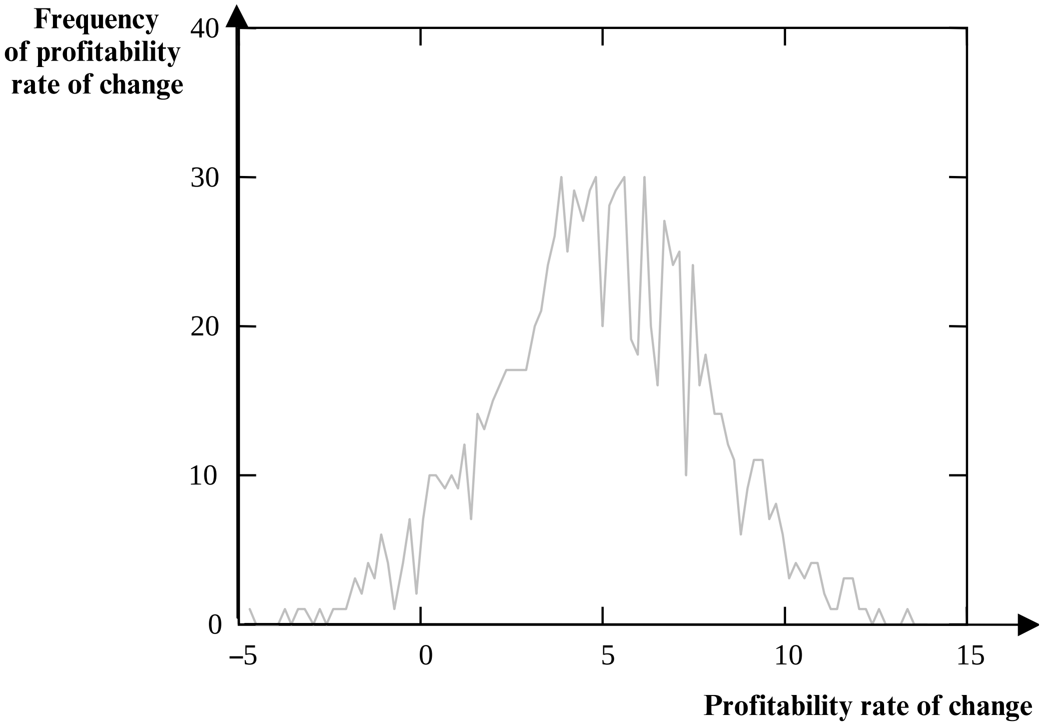


Fig. 26. Frequency of profitability rate of change of the 1-st asset

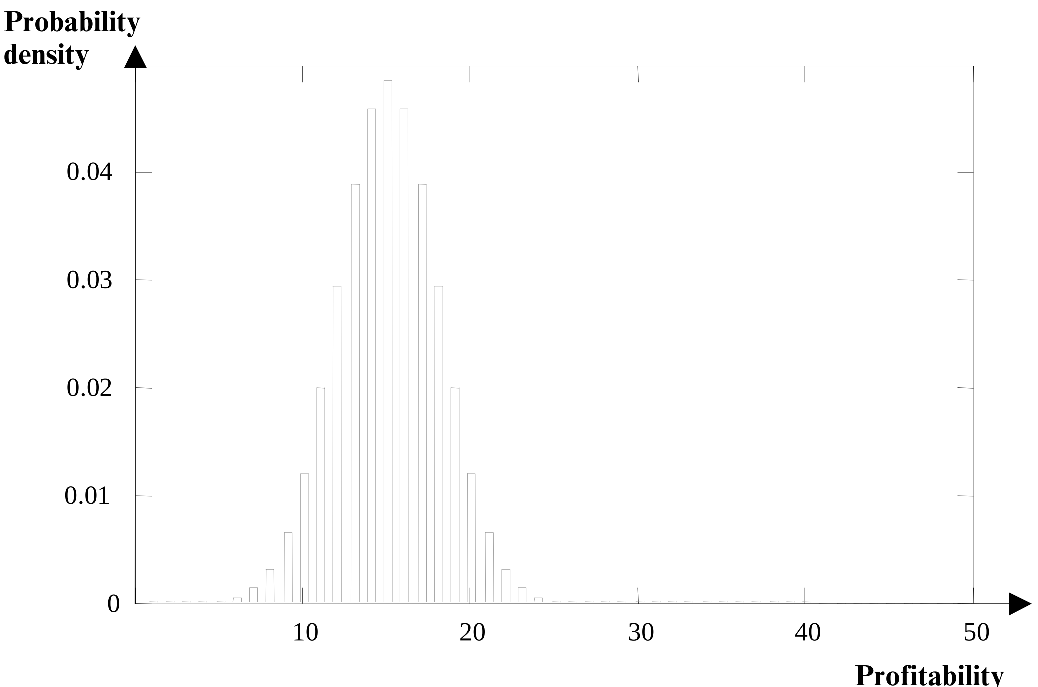


Fig. 27. Probability density of profitability rate of change of the 1-st asset

Fig. 36. Probability density of profitability of the 3-rd asset

2-nd asset:

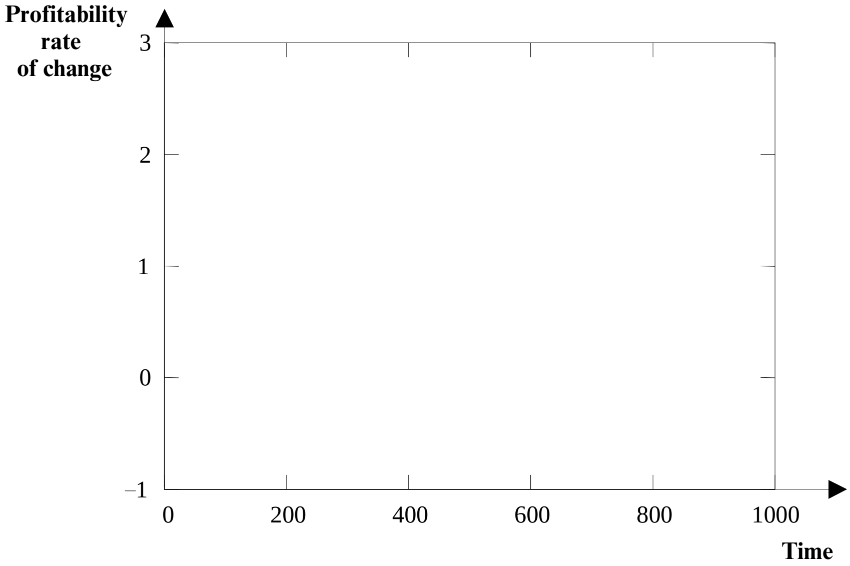


Fig. 37. Profitability rate of change of the 3-rd asset

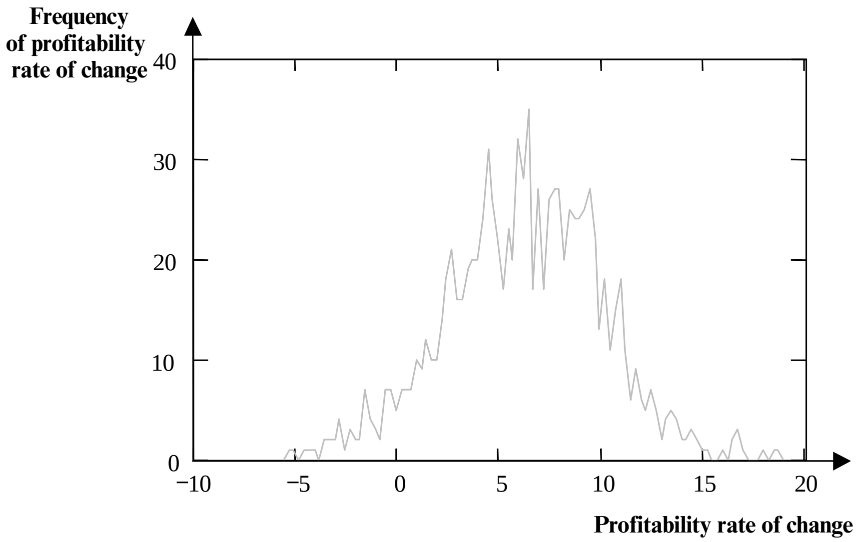


Fig. 38. Frequency of profitability rate of change of the 3-rd asset

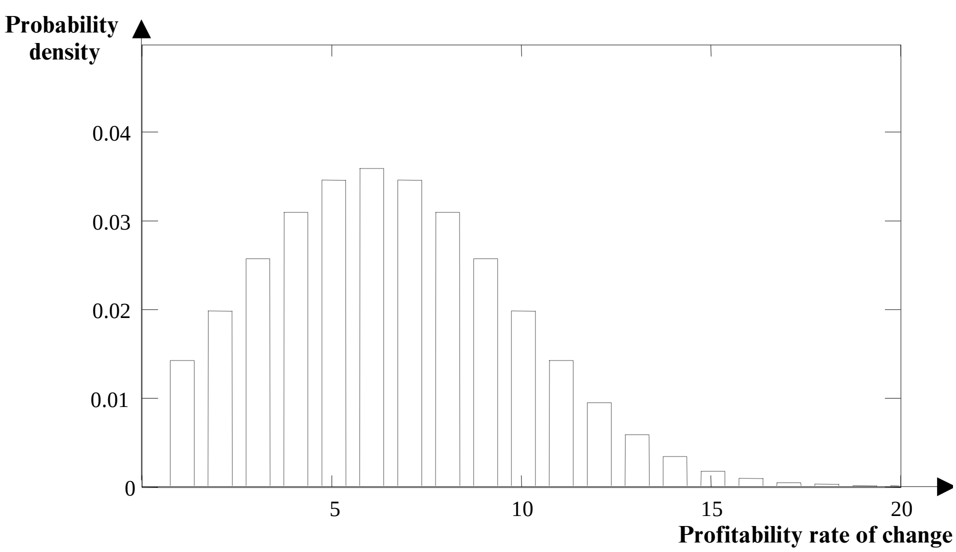


Fig. 39. Probability density of profitability rate of change of the 3-rd asset

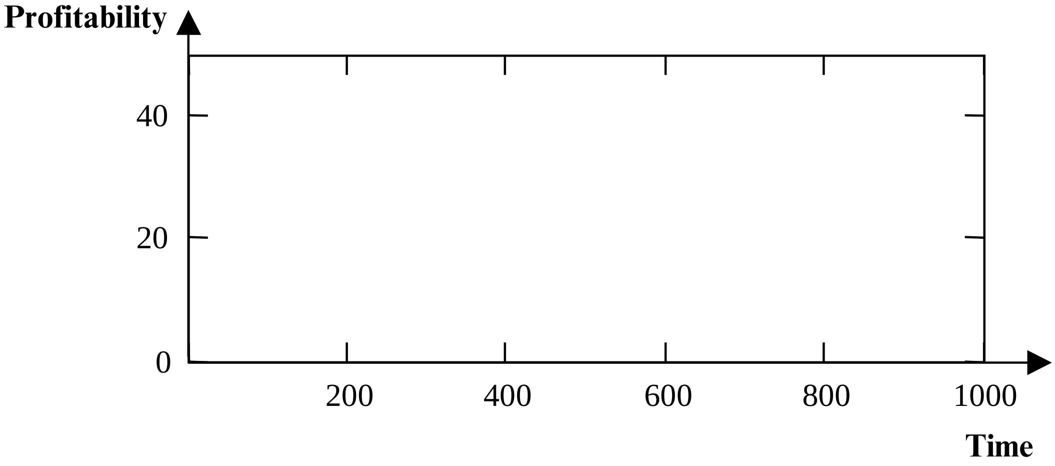


Fig. 28. Dependence of profitability of the 2-nd asset on time

Fig. 40. Dependence of profitability of the 4-th asset on time

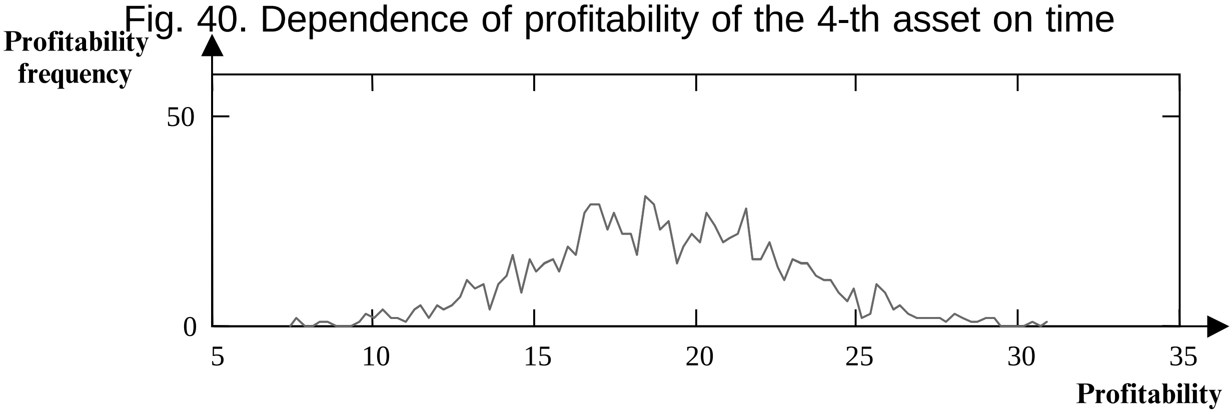


Fig. 29. Frequency of profitability change of the 2-nd asset

Fig. 41. Frequency of profitability change of the 4-th asset

Fig. 30. Probability density of profitability of the 2-nd asset

3-rd asset:

Fig. 42. Probability density of profitability of the 4-th asset

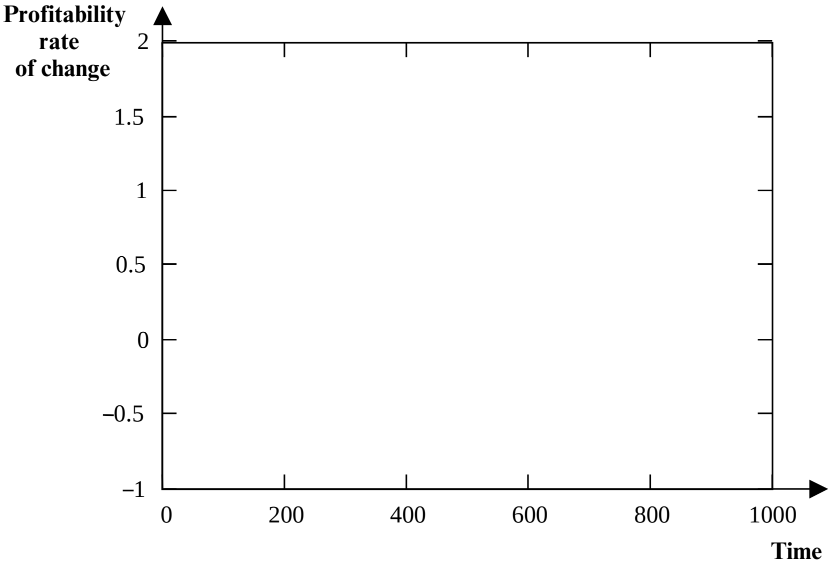


Fig. 43. Profitability rate of change of the 4-th asset

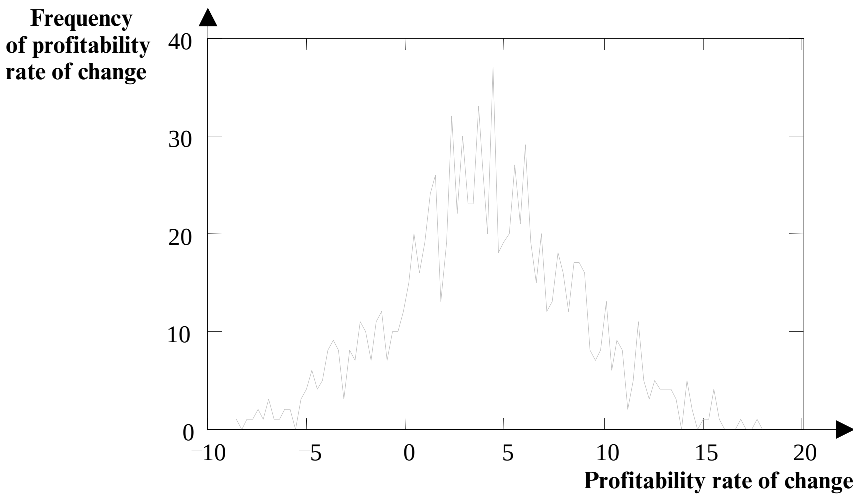


Fig. 44. Frequency of profitability rate of change of the 4-th asset

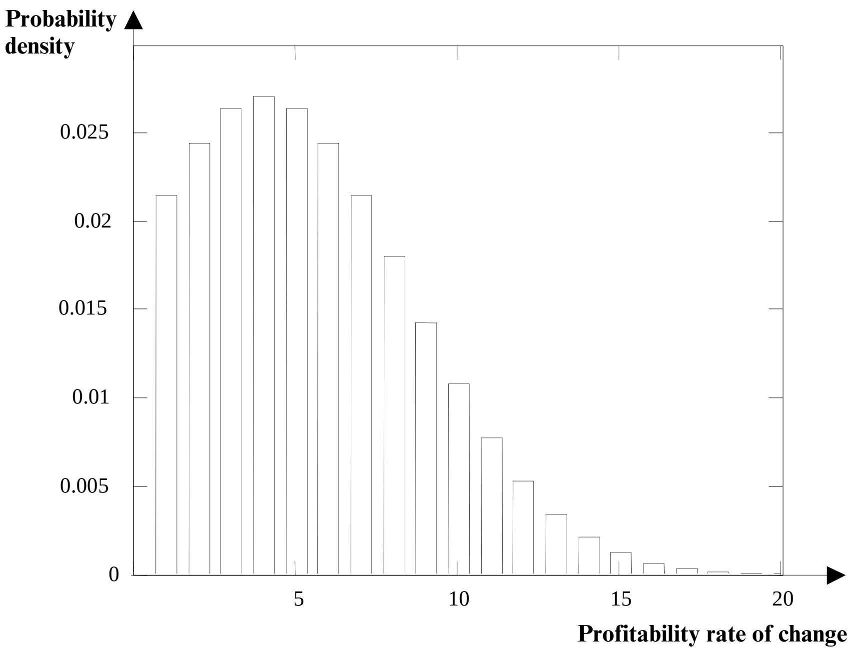


Fig. 45. Probability density of profitability rate of change of the 4-th asset

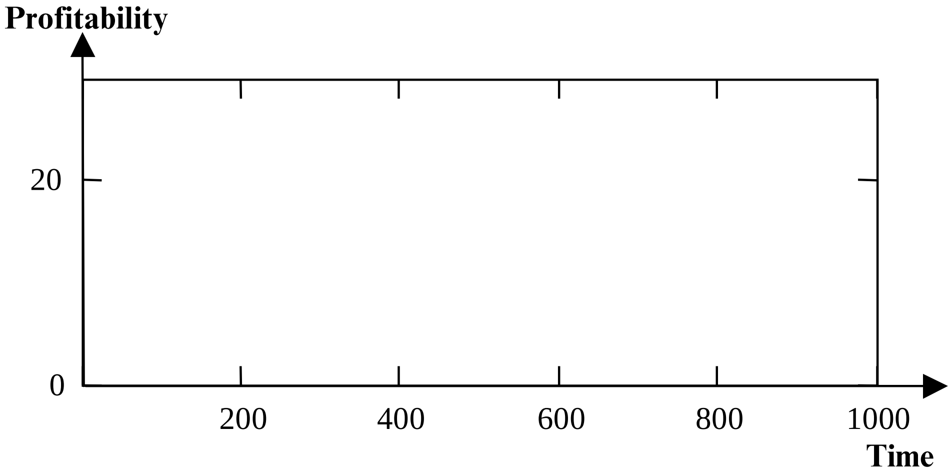


Fig. 46. Dependence of profitability of the 5-th asset on time

Fig. 34. Dependence of profitability of the 3-rd asset on time

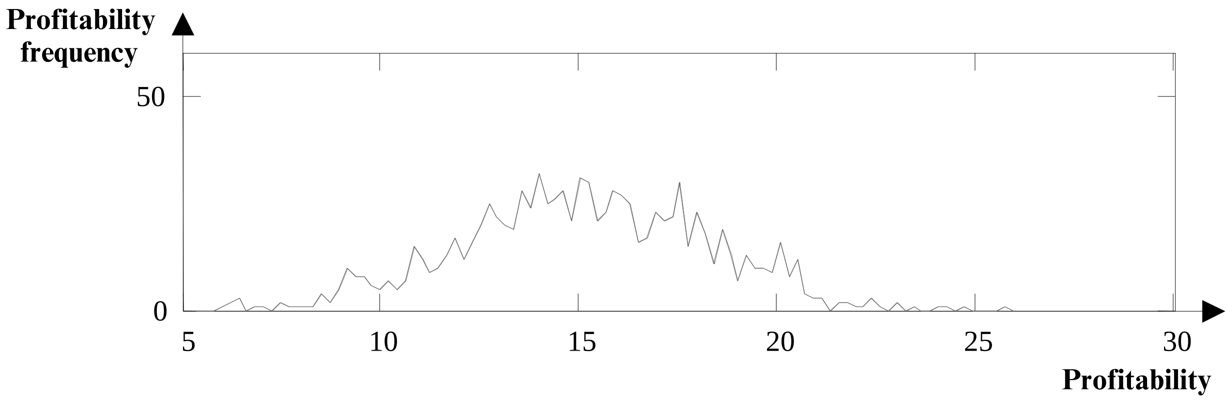


Fig. 47. Frequency of profitability change of the 5-th asset

Fig. 35. Frequency of profitability change of the 3-rd asset

Fig. 48. Probability density of profitability of the 5-th asset

4-th asset:

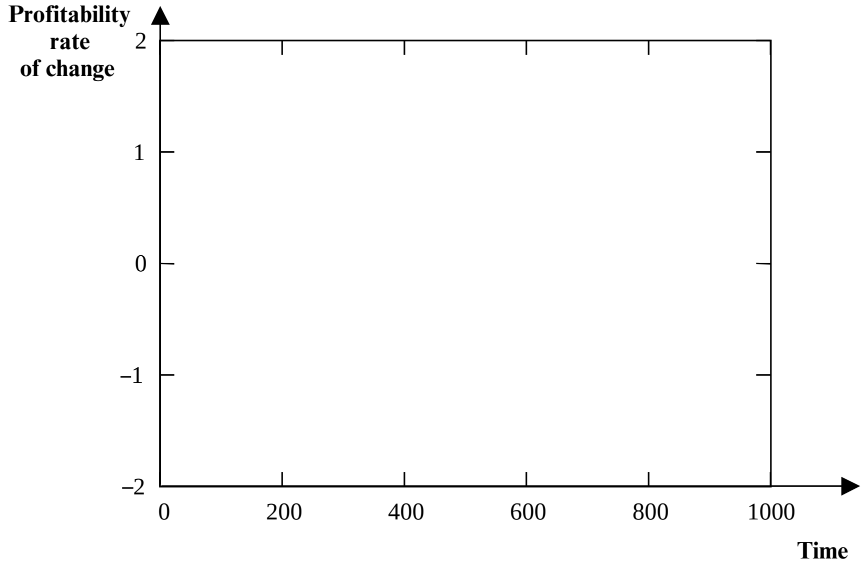


Fig. 49. Profitability rate of change of the 5-th asset

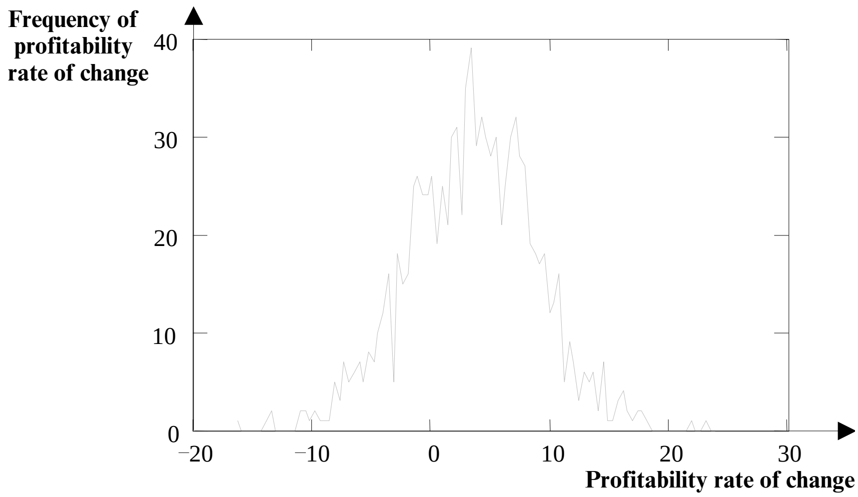


Fig. 50. Frequency of profitability rate of change of the 5-th asset

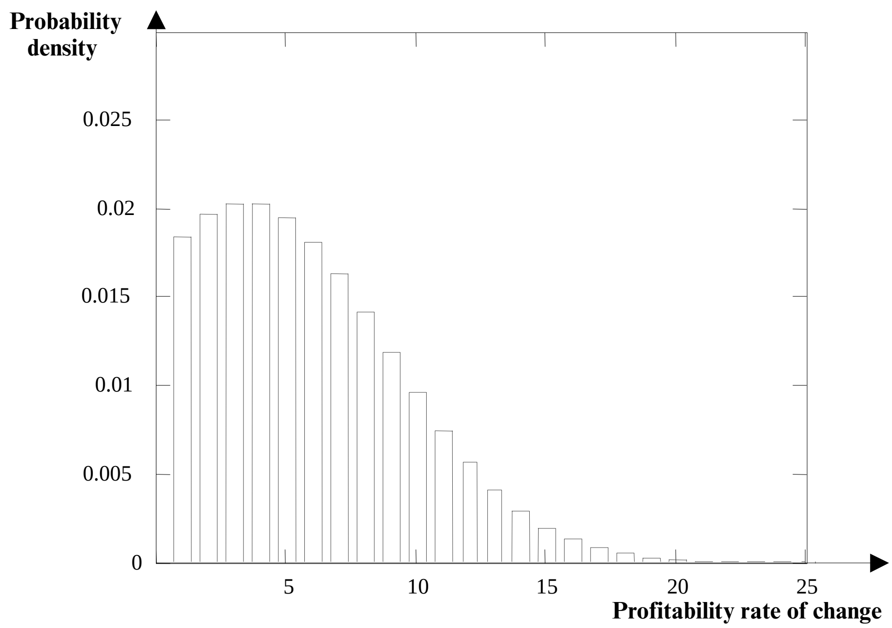


Fig. 51. Probability density of profitability rate of change of the 5-th asset

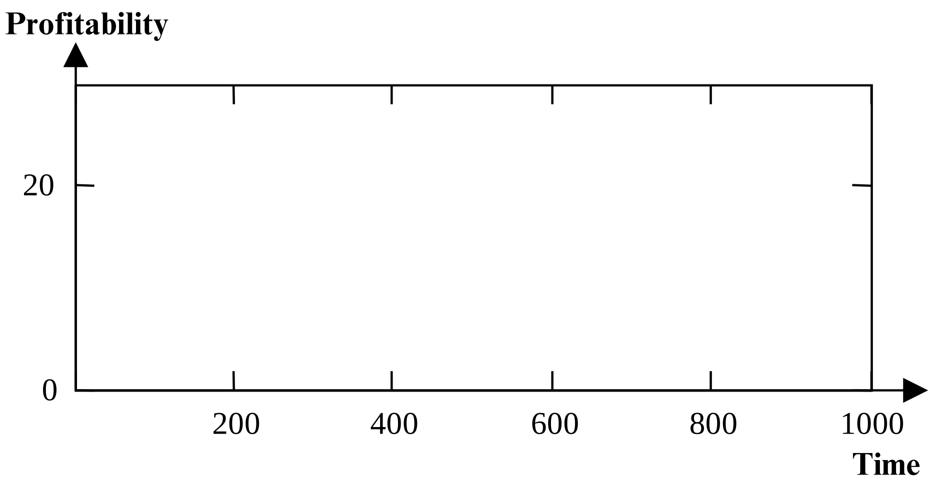


Fig. 52. Dependence of profitability of the 6-th asset on time

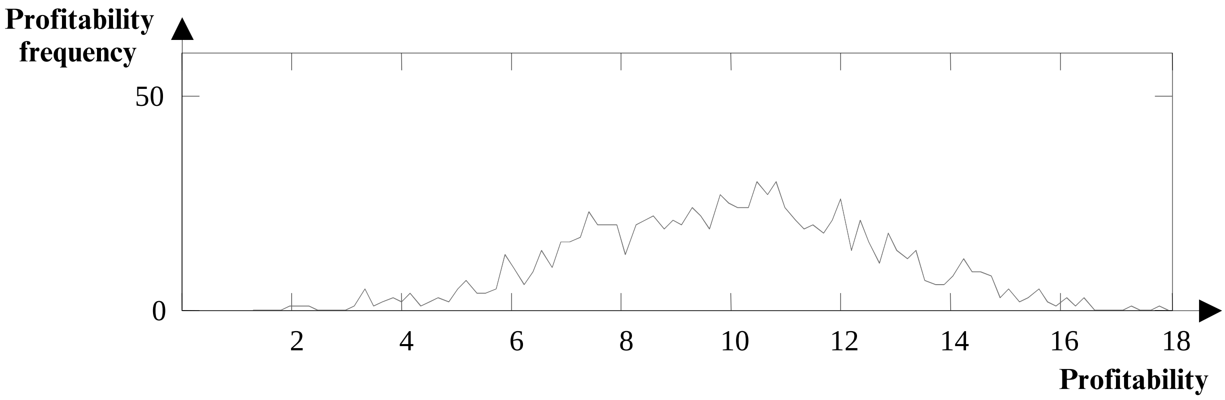


Fig. 53. Frequency of profitability change of the 6-th asset

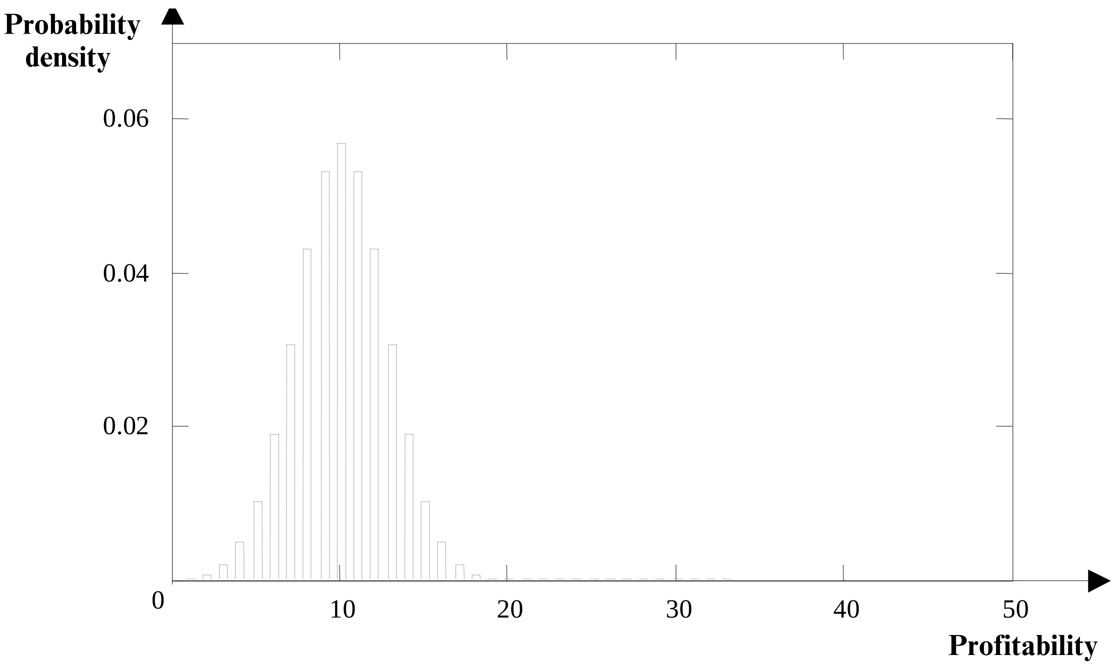


Fig. 54. Probability density of profitability of the 6-th asset

5-th asset:

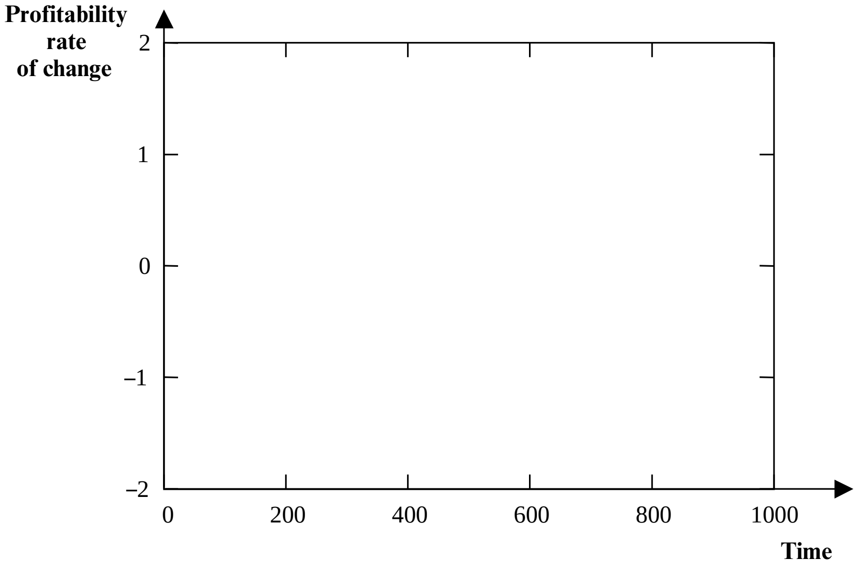


Fig. 55. Profitability rate of change of the 6-th asset

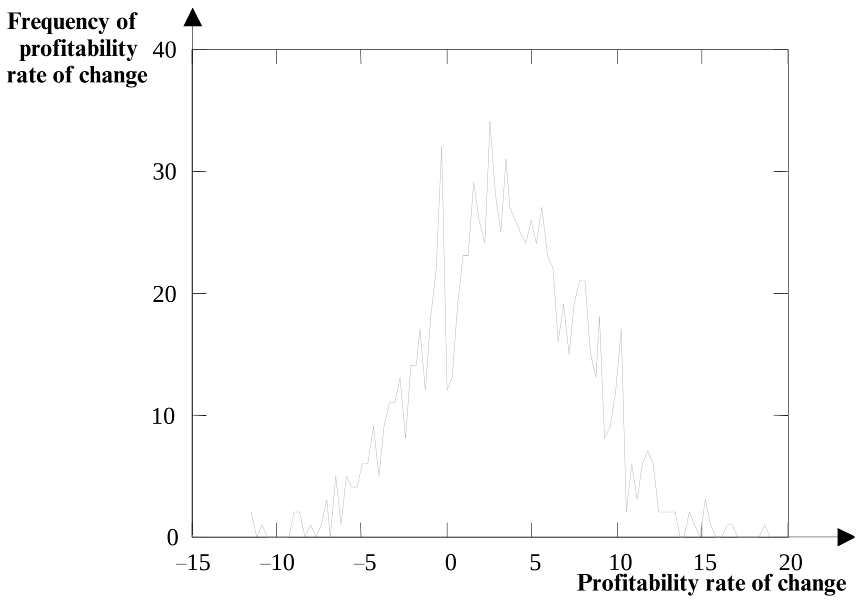


Fig. 56. Frequency of profitability rate of change of the 6-th asset

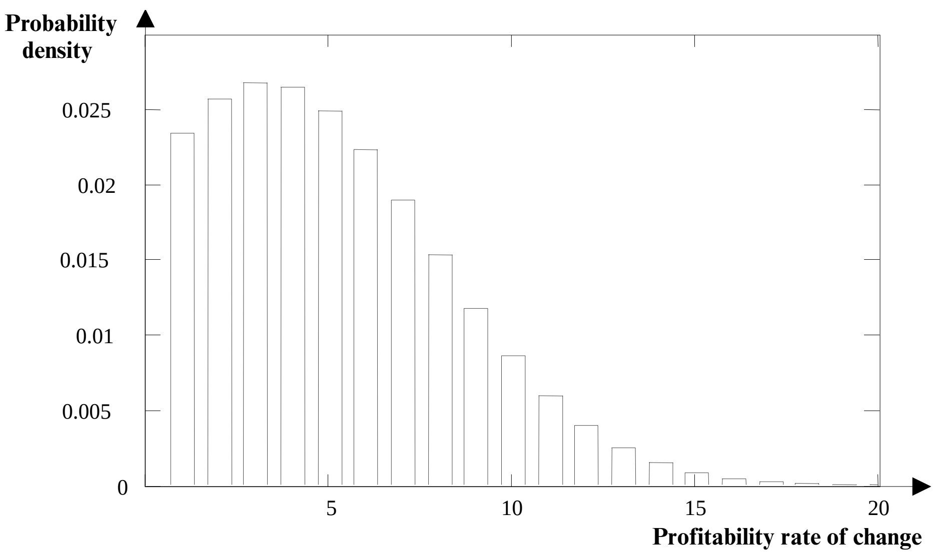


Fig. 57. Probability density of profitability rate of change of the 6-th asset

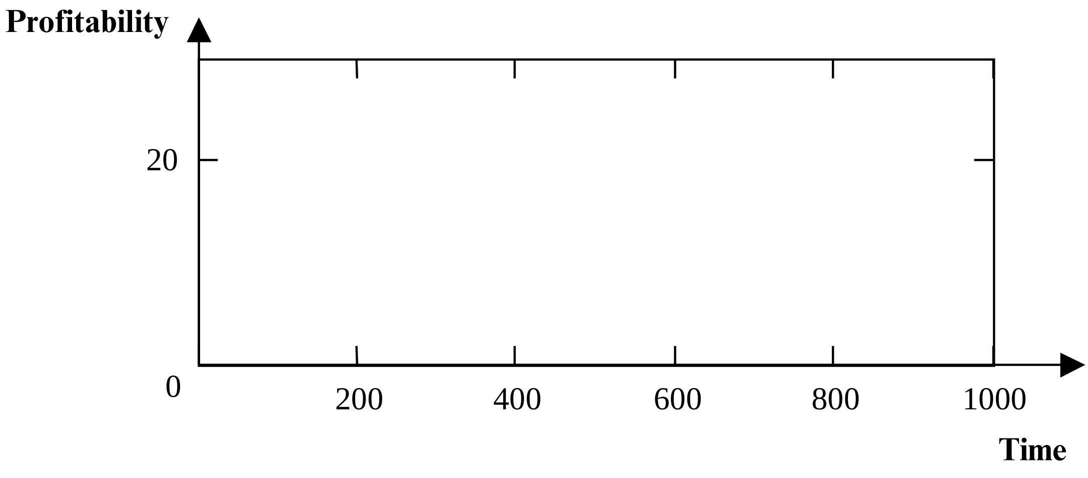


Fig. 58. Dependence of profitability of the 7-th asset on time

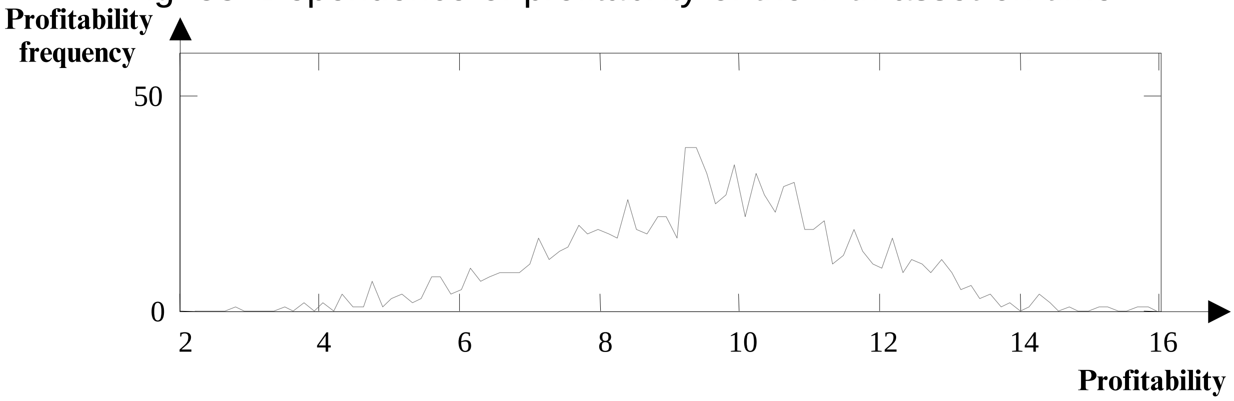


Fig. 59. Frequency of profitability change of the 7-th asset

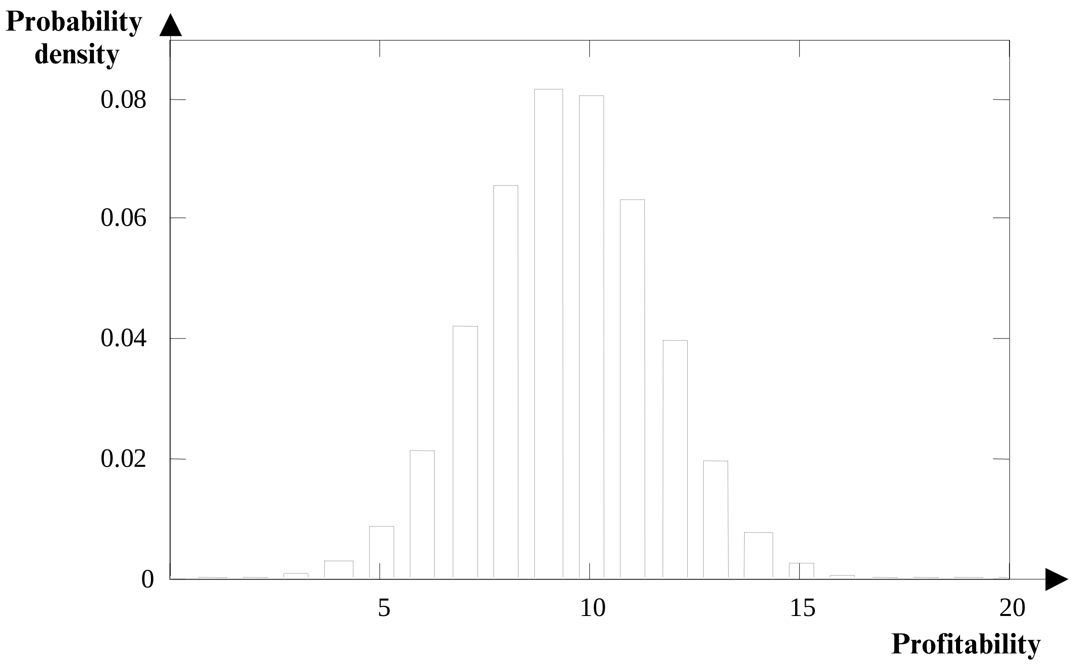


Fig. 60. Probability density of profitability of the 7-th asset

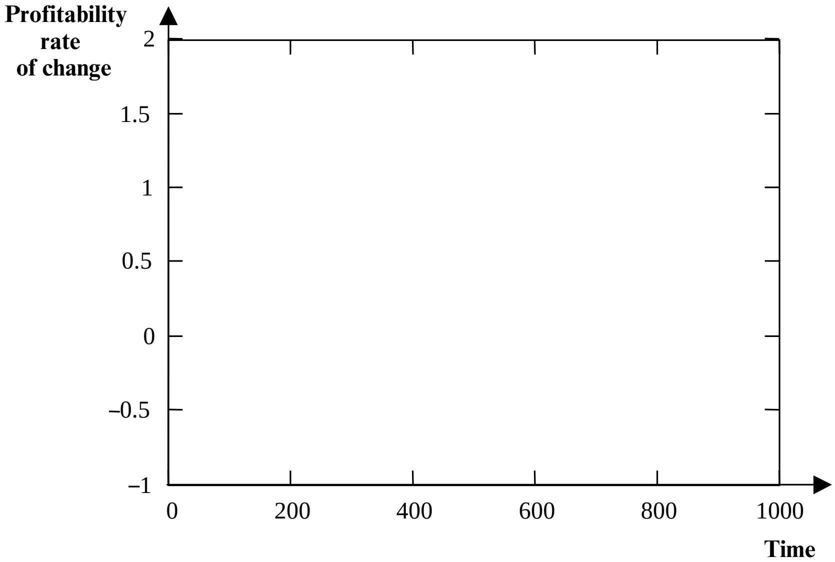


Fig. 61. Profitability rate of change of the 7-th asset

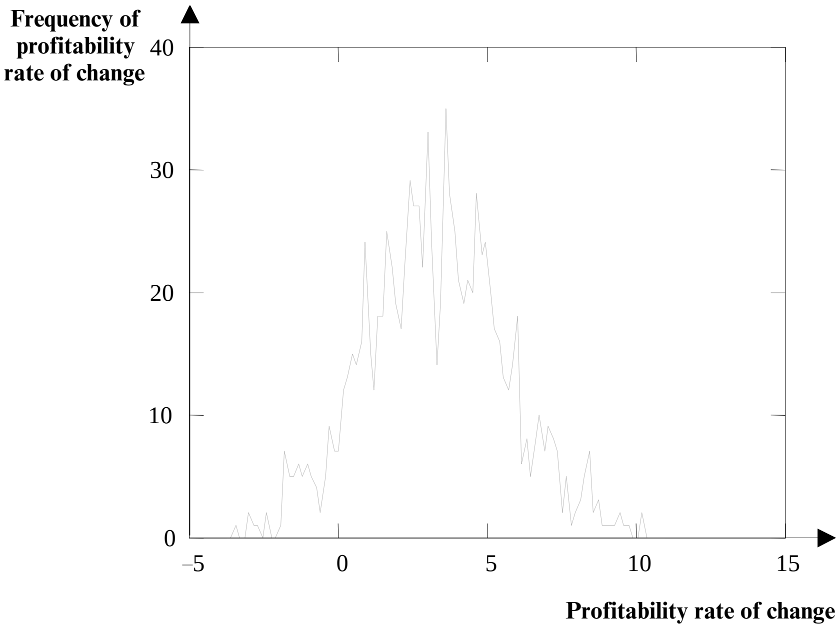


Fig. 62. Frequency of profitability rate of change of the 7-th asset

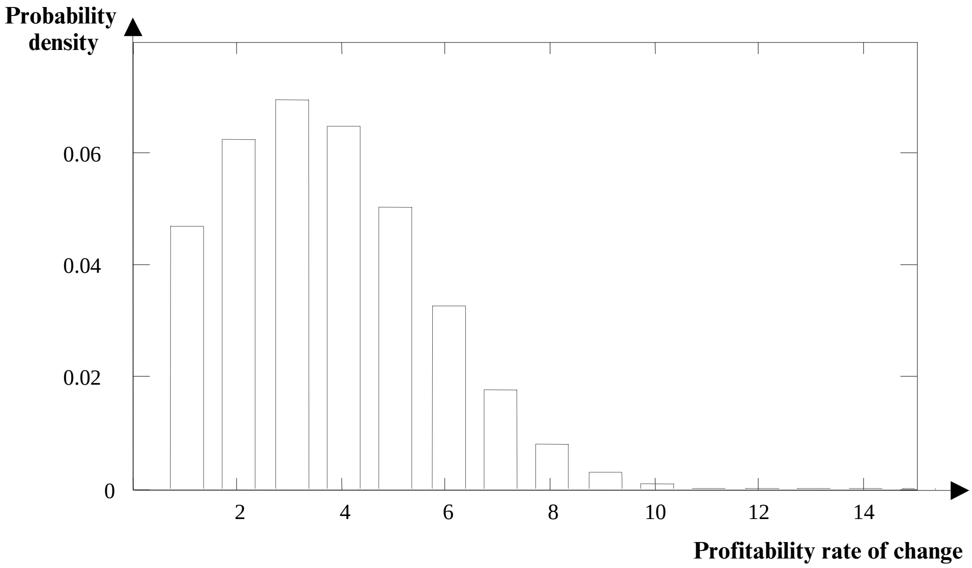


Fig. 63. Probability density of profitability rate of change of the 7-th asset

6-th asset:

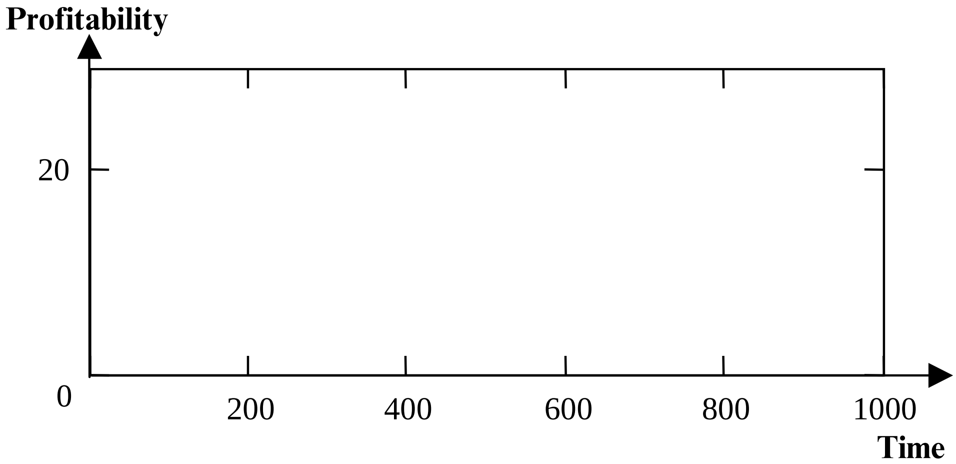


Fig. 64. Dependence of profitability of the 8-th asset on time

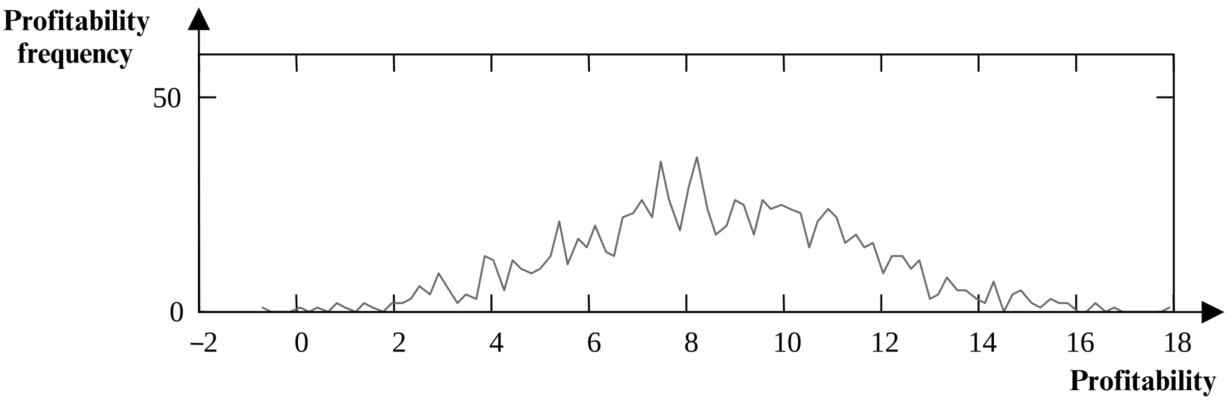


Fig. 65. Frequency of profitability change of the 8-th asset

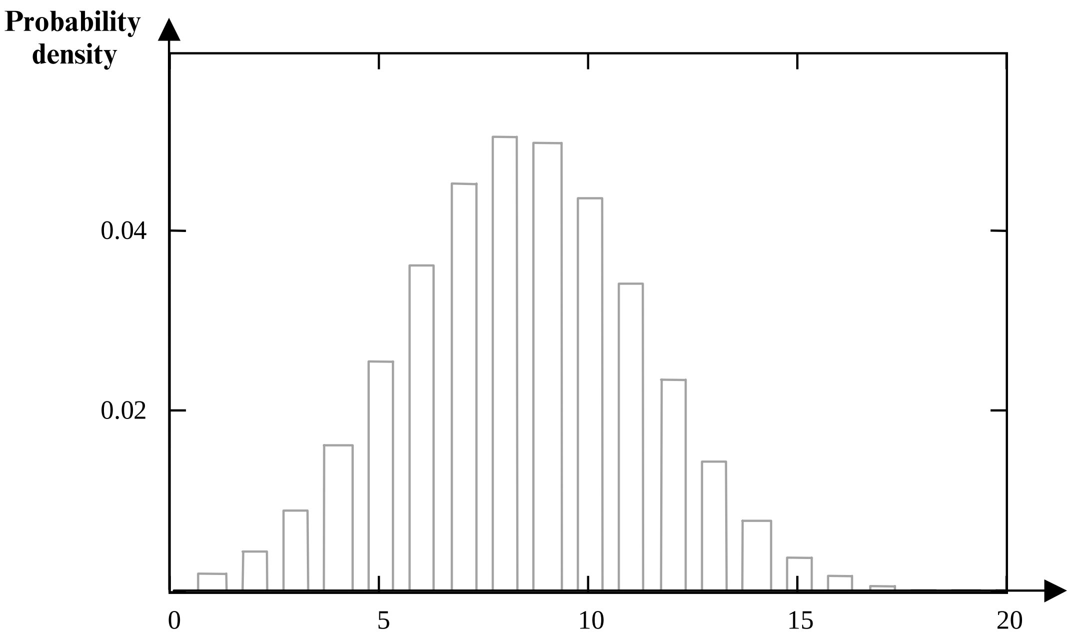


Fig. 66. Probability density of profitability of the 8-th asset

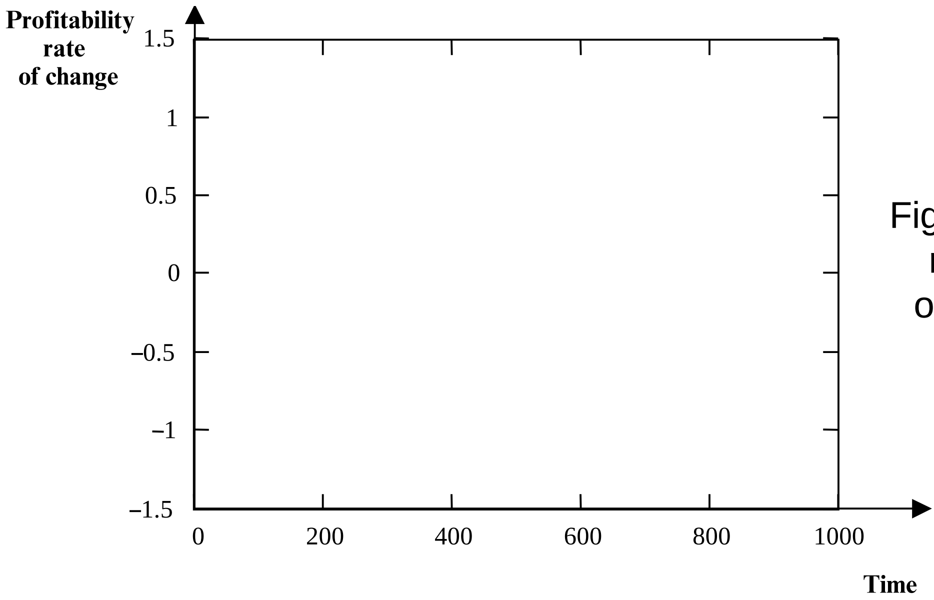


Fig. 67. Profitability rate of change of the 8-th asset

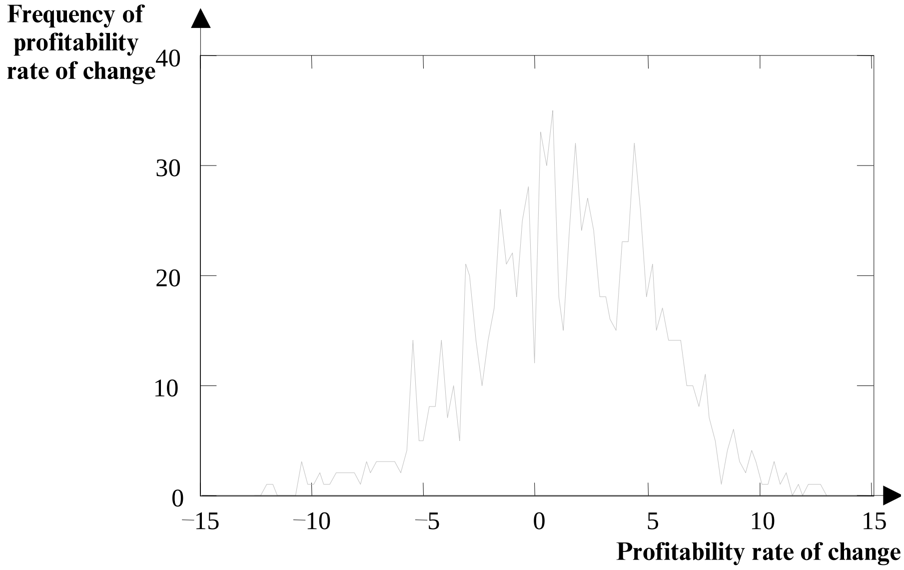


Fig. 68. Frequency of profitability rate of change of the 8-th asset

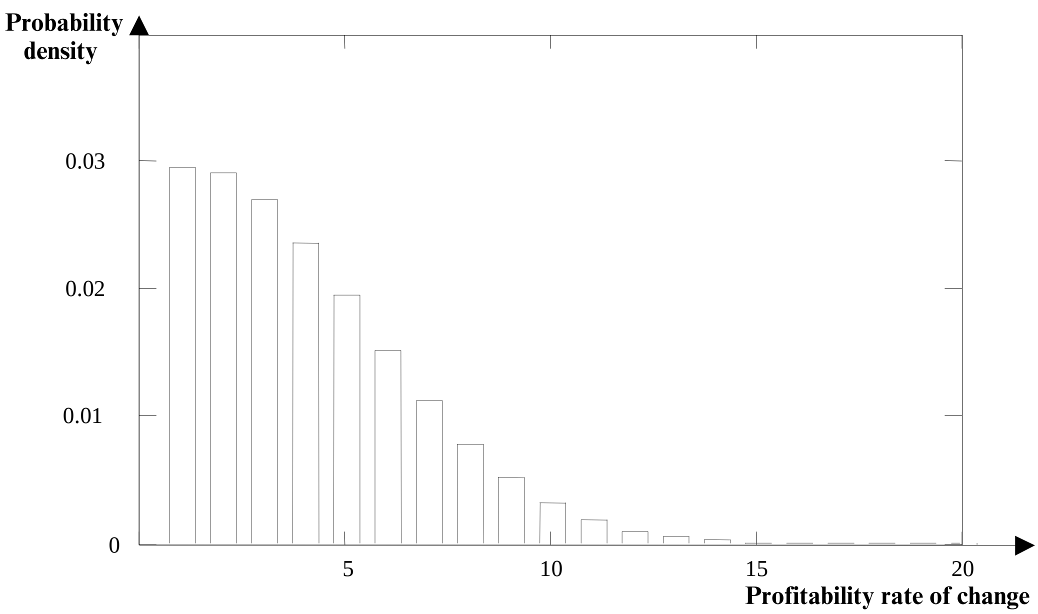


Fig. 69. Probability density of profitability rate of change of the 8-th asset

7-th asset:

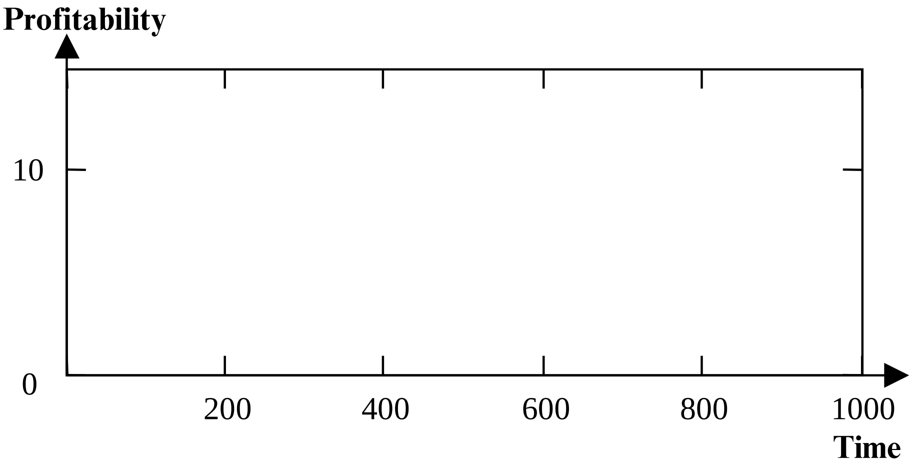


Fig. 70. Dependence of profitability of the 9-th asset on time

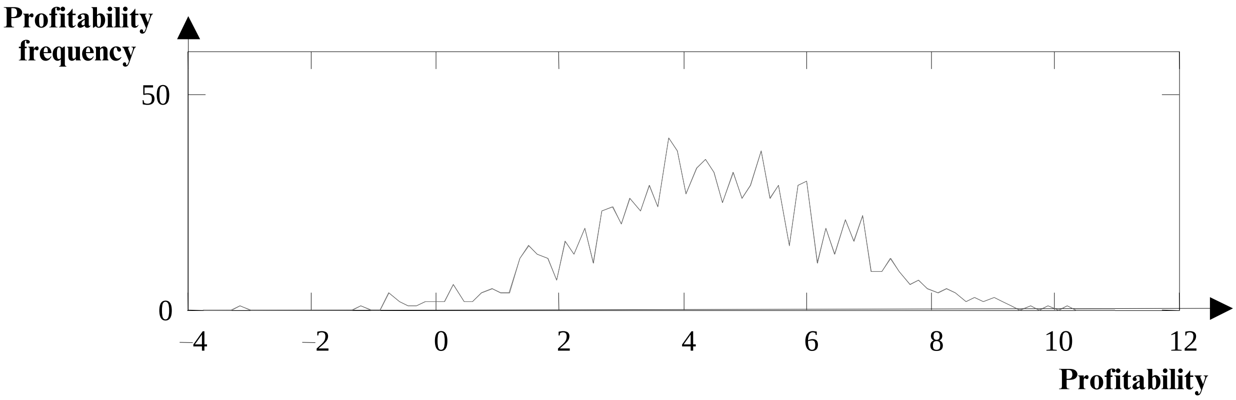


Fig. 71. Frequency of profitability change of the 9-th asset

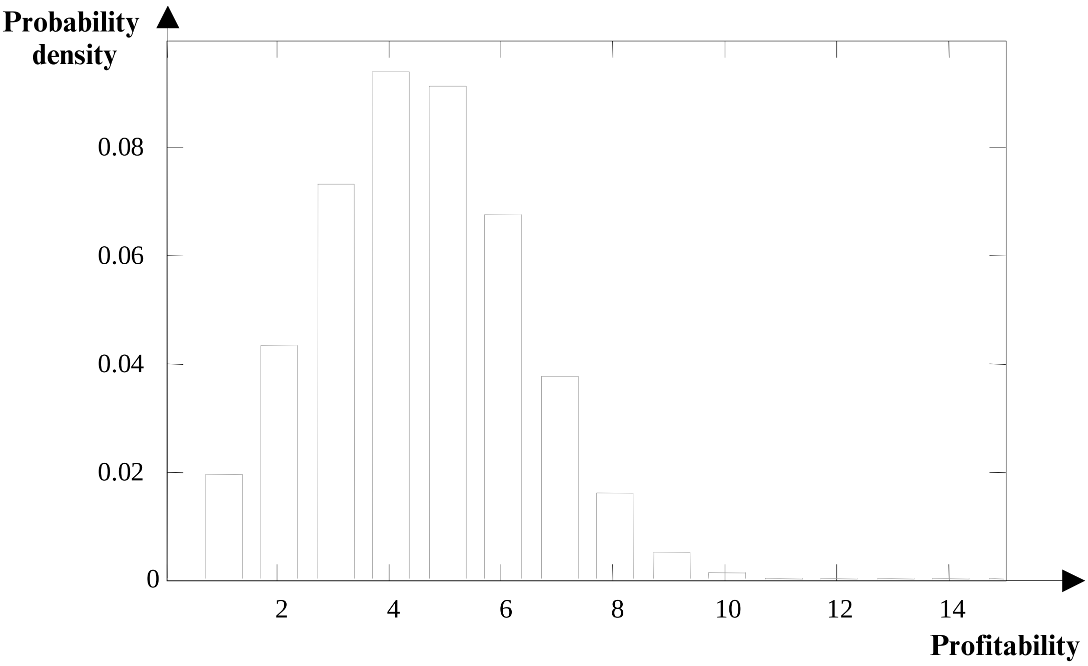


Fig. 72. Probability density of profitability of the 9-th asset

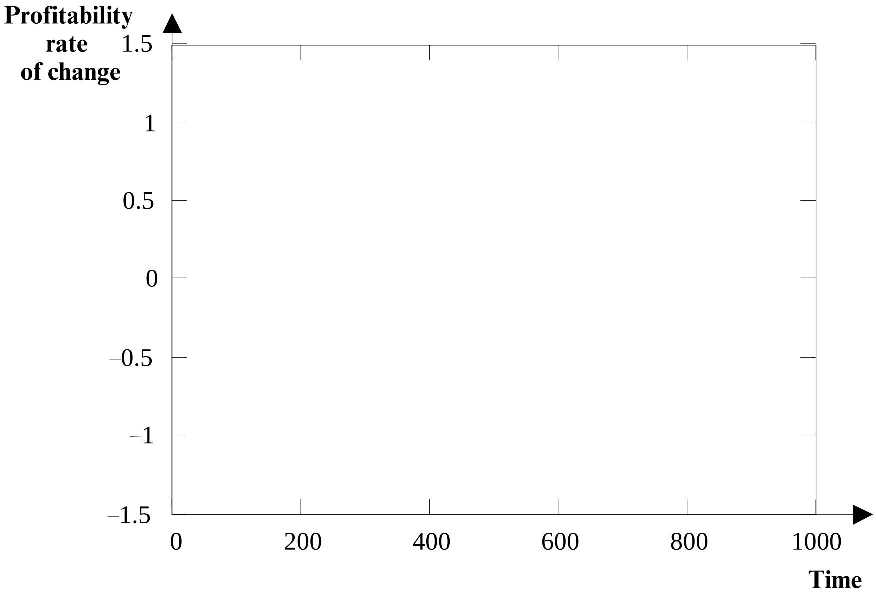


Fig. 73. Profitability rate of change of the 9-th asset

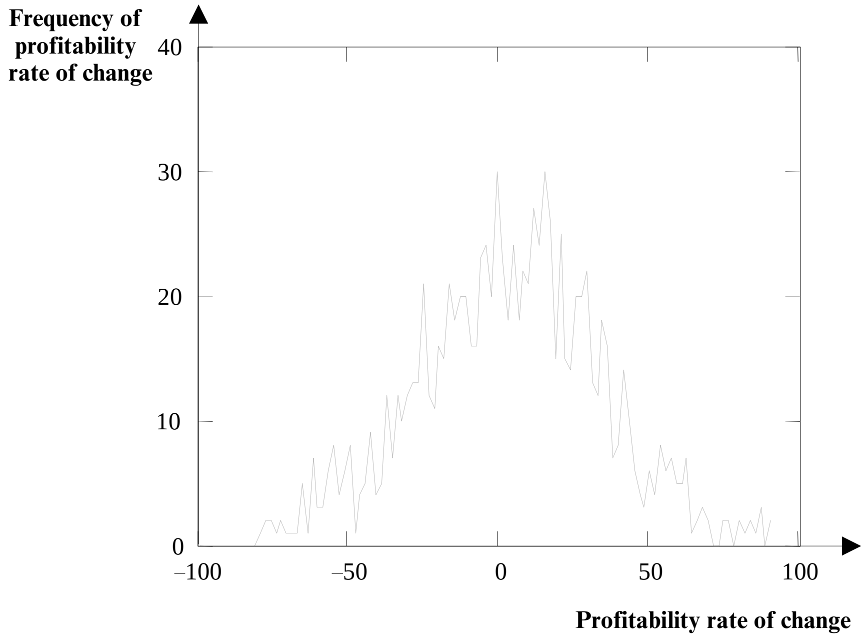


Fig. 74. Frequency of profitability rate of change of the 9-th asset

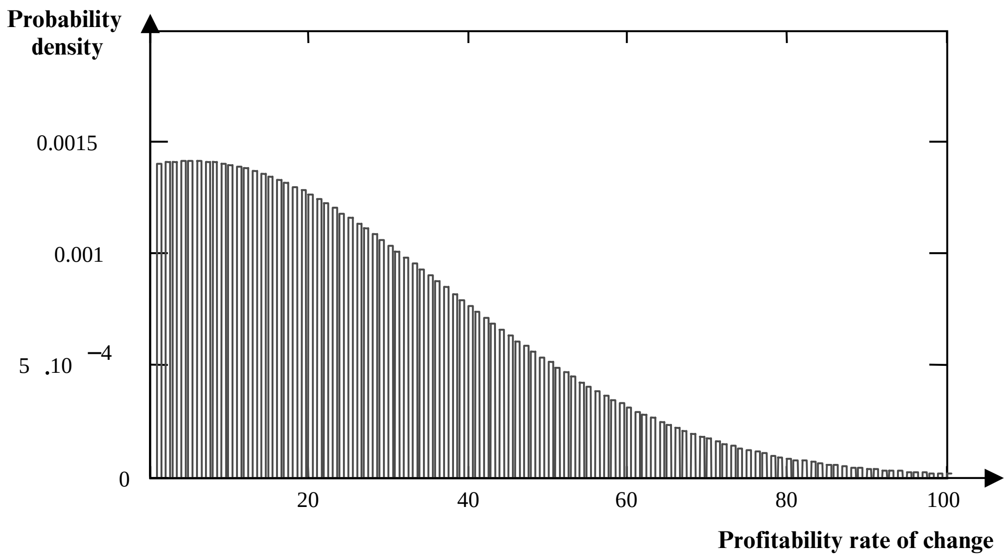
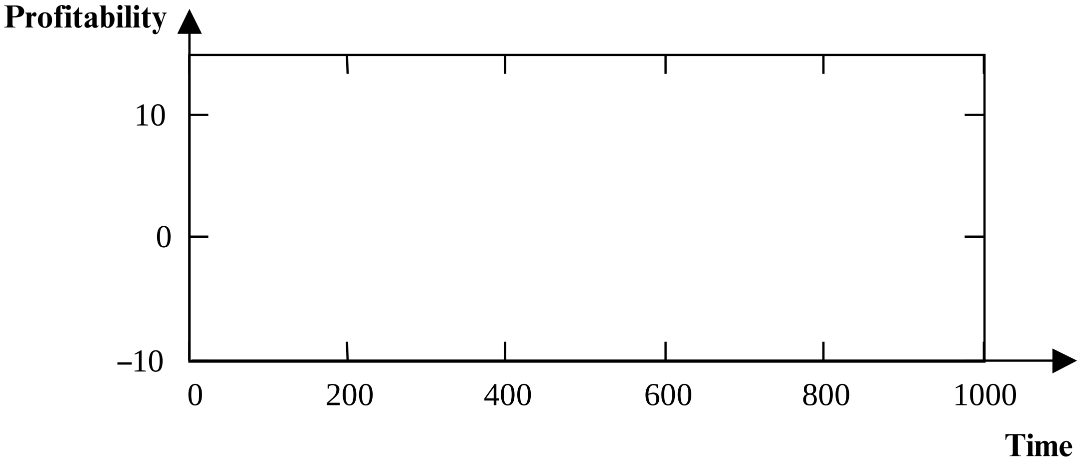
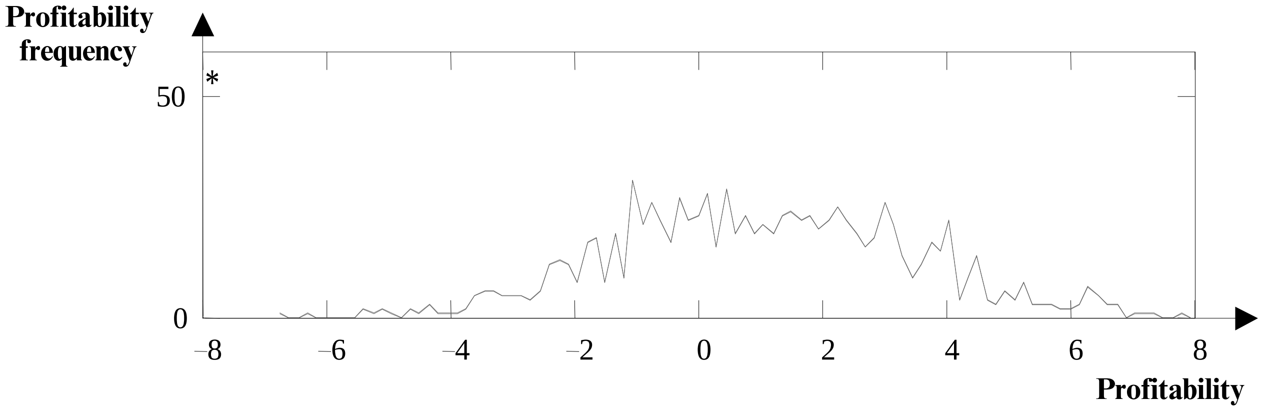


Fig. 75. Probability density of profitability rate of change of the 9-th asset

8-th asset:



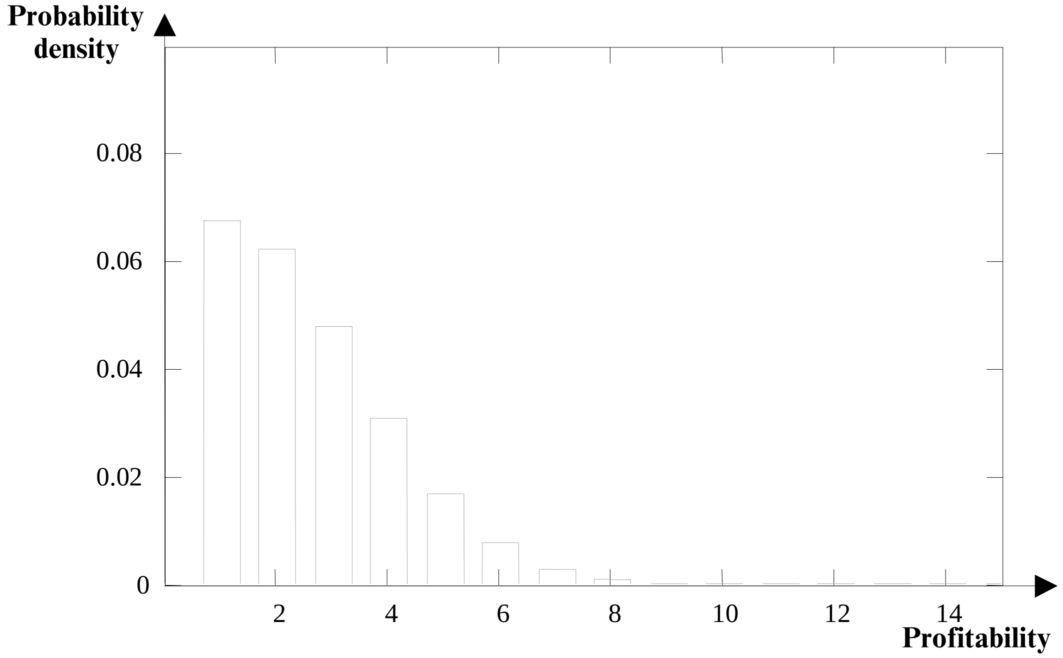
Profitability
10
0
–10
0
200
400
600
800
1000
Time



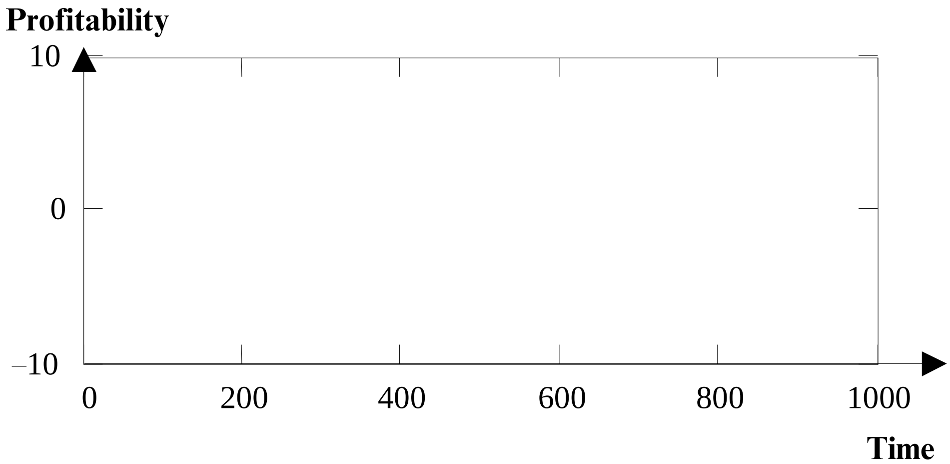
Profitability
frequency
50
0
*
–8
–6
–4
–2
0
2
4
6
8
Profitability

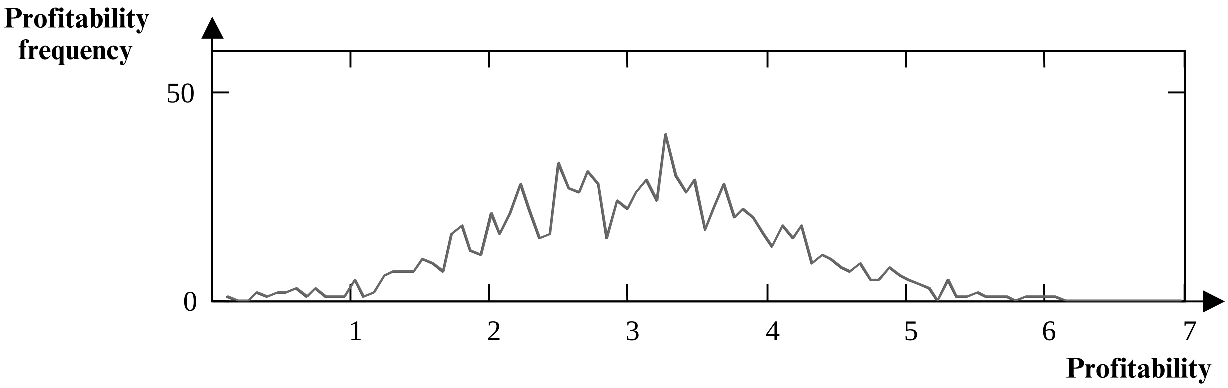
$$R_{ij}^{2} = \begin{pmatrix}
-5.622\cdot 10^{3} & -1.765\cdot 10^{-5} & 1.248\cdot 10^{3} & 206.116 & 493.995 & -359.237 & 81.695 & 16.912 & 16.557 \\
-1.003\cdot 10^{5} & -7.2\cdot 10^{3} & -39.108 & -781.88 & 670.021 & -929.987 & 58.823 & 9.991 & 0.152 \\
1.248\cdot 10^{3} & -39.107 & -8.55\cdot 10^{3} & 0.075 & 750.051 & -92.136 & -14.58 & -0.713 & 1.179 \\
206.099 & -781.865 & 0.075 & -1.036\cdot 10^{4} & -1.401\cdot 10^{3} & 1.441\cdot 10^{-6} & -3.736\cdot 10^{-3} & 0.099 & -0.173 \\
493.978 & 670.069 & 750.041 & -1.401\cdot 10^{3} & -9.149\cdot 10^{3} & -1.142\cdot 10^{3} & -4.731\cdot 10^{-5} & -2.331\cdot 10^{-4} & -2.791\cdot 10^{-4} \\
-359.357 & -929.993 & -92.132 & -4.33\cdot 10^{-7} & -1.142\cdot 10^{3} & -7.476\cdot 10^{3} & 1.319 & 1.229 & 1.408 \\
81.692 & 58.817 & -14.58 & -3.661\cdot 10^{-3} & 1.784\cdot 10^{-5} & 1.319 & -2.116\cdot 10^{3} & 2.169 & 1.705 \\
16.91 & 9.991 & -0.713 & 0.099 & -2.163\cdot 10^{-4} & 1.229 & 2.169 & -626 & 8.875\cdot 10^{-5} \\
16.557 & 0.152 & 1.179 & 0.173 & 2.84\cdot 10^{-4} & 1.408 & 1.705 & 3.253\cdot 10^{-4} & 950
\end{pmatrix}$$

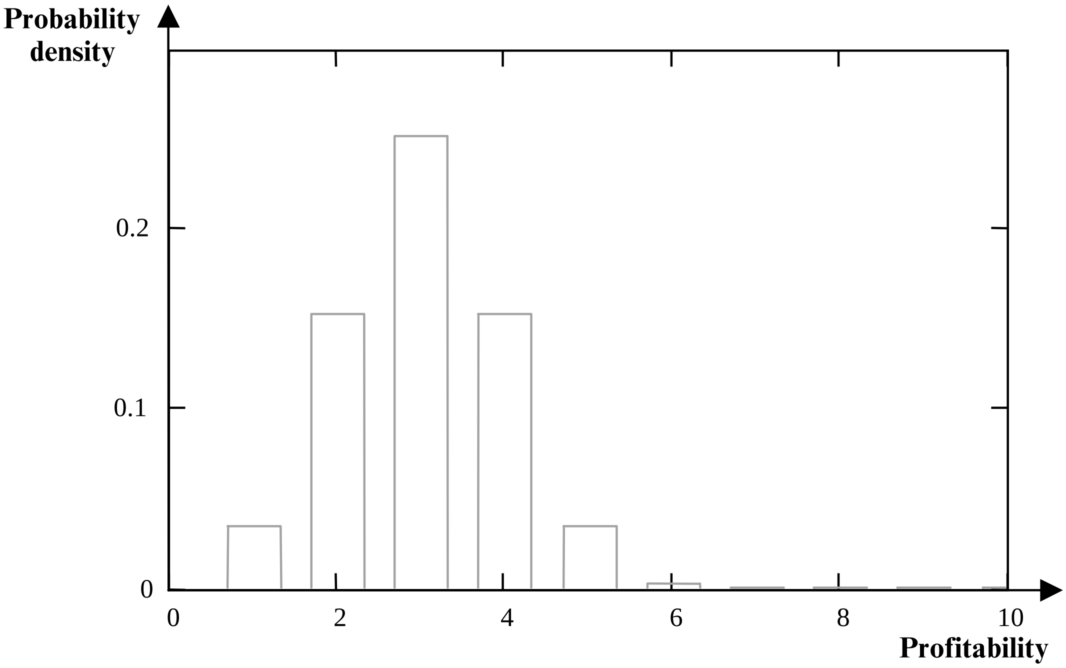
Fig. 76. Eigenvalues of the matrix R_{ij}^{2}


Probability density
0.08
0.06
0.04
0.02
0
2
4
6
8
10
12
14
Profitability

9-th asset:


Profitability
10
0
–10
0
200
400
600
800
1000
Time





Let's write down equations for the matrix elements R_{ij}^2. We shall presume that h = 0.1, γ=1.

Taking into account that distribution of profitability and PRC is in the given case follow the normal law, we have:

Mathematical supplement. The generalized Hermitian task about eigenvalues as the variational task

For the set of linear boundary conditions, **the eigenfunction (characteristic function)** of the linear differential operator L is the solution $\psi(x)$ that is not identically equal to zero in V of the differential equation

$$L\psi(x) = \lambda\psi(x) \qquad (x \in V), \tag{S.1}$$

where λ – a certain number appropriately defined and called **as the eigenvalue (characteristic number) of the operator** L that is connected with the eigenfunction $\psi(x)$.

A more common task on eigenvalues is the task on determination the eigenfunctions $\psi(x) \neq 0$ and the eigenvalues λ, which satisfy the differential equation

$$L\psi(x) = \lambda B(x)\psi(x) \qquad (x \in V) \tag{S.2}$$

and the given linear uniform boundary conditions; B(x) is the real and positive function in V.

If the operator L in the equation (S.1) is Hermite, then

1) all values λ of the spectrum are real;

2) the normalized eigenfunctions ψ_i, ψ_k, which correspond to various eigenvalues, are mutually orthogonal:

$$(\psi_i, \psi_k) = \int_V \overline{\psi}_i \psi_k dV = 0 \qquad (i \neq k).$$

The task on eigenvalues (S.2) for the Hermitian of the differential operator L with discrete eigenvalues $\lambda_1, \lambda_2, ...$ is equivalent to each of the following variational tasks:

1. To find the function $\psi(x) \neq 0$ in V, that would satisfy the given boundary conditions and turn the variation of the functional into zero:

$$\frac{(\psi, L\psi)}{(\psi, B\psi)} = \frac{\int_V \overline{\psi} L\psi dV}{\int_V |\psi|^2 B dV} \qquad \text{(quotient of Reley)}. \tag{S.3}$$

2. To find the function $\psi(x)$ that would satisfy the given boundary conditions and turn the variation of the functional

$$(\psi, L\psi) = \int_V \overline{\psi} L\psi dV$$

into zero under condition that:

$$(\psi, B\psi) = \int_V |\psi|^2 BdV = 1.$$

In each of these cases, the function $\psi = \psi_k(x)$ corresponds to the stationary value of the functional λ_k.

Thus, application of the direct approach of calculus of variations, in particular, the method of Reley-Ritz is possible for solving the task on eigenvalues for the ordinary differential equations and the equations with partial derivatives.

Let's presume that the Hermitian operator L with the discrete spectrum that contains not more then the finite number of negative eigenvalues is given, and let's presume that the eigenvalues are arranged in their ascending order, and the eigenvalues of the class m are repeated m times: $\lambda_1 \le \lambda_2 \le ...$ The least eigenvalue λ_1 is equal to the minimum quotient of Reley (S.3) for arbitrary "admissible"\ functions, which satisfy the given boundary conditions and those, which allow the quotient of Reley to exist.

Similarly the r-eigenvalue λ_r of the above mentioned sequence does not exceed the quotient of Reley for all "admissible" functions $\psi(x)$, so that

$$\int_V \overline{\psi}_k \varphi BdV = 0$$

for each eigenfunction ψ_k, corresponding to $\lambda_1, \lambda_2, ..., \lambda_{r-1}$ (the minimax principle of Courant).

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